Write your name below. This exam is worth 100 points and has 6 questions. On each problem, you must show all your work to receive credit on that problem. You are allowed to use one page of notes (one sided). You cannot collaborate on the exam or seek outside help, nor can you use the recorded lectures, a calculator, any computational software, or material you find online.

Name:

1. (21 points, 7 each) In each of the following problems, either show that the statement is always true or provide a counterexample that shows it is false.

   (a) The function
   $$p(\mathbf{x}) = x^2 - 2xz + y^2 - 2yz + 2z^2 - 2x - 2y - 2z + 4$$
   does not have a minimum value on $\mathbb{R}^3$.

   (b) If A^+ is the pseudoinverse of A then $(A^+A)^T = A^+A$.

   (c) The quadratic form $q(\mathbf{v}) = \mathbf{v}^T K \mathbf{v}$ defines a linear transformation $Q : \mathbb{R}^n \to \mathbb{R}$.

2. (20 points) Let A $= \begin{pmatrix} -1 & 0 & 3 \\ 3 & 2 & -2 \\ 0 & 0 & 2 \end{pmatrix}$

   (a) (10 points) Find the eigenvectors of A.

   (b) (3 points) Is A diagonalizable? Why or why not?

   (c) (7 points) If A is diagonalizable, find the matrices S and Λ that diagonalize it. If A is **not** diagonalizable, find the Jordan Canonical Form J and matrix S such that $A = SJS^{-1}$. You do not need to calculate S^{-1} in either case.

3. (20 points) Let

$$A = \begin{pmatrix} 1 & 2 \\ -3 & -1 \\ -1 & 3 \end{pmatrix}$$

(a) (4 points) Show that A has singular values $\sigma_1 = \sqrt{15}$ and $\sigma_2 = \sqrt{10}$.

(b) (7 points) Find the best rank 1 approximation of A.

(c) (9 points) Find the pseudoinverse of A.

4. (20 points) Let L be the linear function $L : \mathbb{R}^3 \to \mathbb{R}^2$ that is given in the standard basis by

$$L \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + 2y + 3z \\ y + 2z \end{pmatrix}$$

For all of the questions below, we wish to find the bases for $\mathbb{R}^3$ and $\mathbb{R}^2$ that put L into the canonical form.

(a) (2 points) Find the matrix representation of L in the standard basis.

(b) (8 points) What basis should we choose for $\mathbb{R}^3$?

(c) (6 points) What basis should we choose for $\mathbb{R}^2$?

(d) (4 points) Verify that your bases are the correct bases.

5. (10 points) Let A be a symmetric 2×2 matrix with eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$ and eigenvalue λ_1 has the eigenvector $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

   (a) (5 points) Find A.

   (b) (5 points) Find the matrix exponential e^{At}.

6. (9 points) Let

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{pmatrix} 3 \\ 3 \\ 0 \\ 0 \end{pmatrix}$$

Find the vector in img$A$ that is closest to $\mathbf{b}$ with the standard dot product.