Write your name below. This exam is worth 100 points and has 5 questions. On each problem, you must show all your work to receive credit on that problem. You are allowed to use one page of notes (one sided). You cannot collaborate on the exam or seek outside help, nor can you use the recorded lectures, a calculator, any computational software, or material you find online.

Name:

1. (21 points, 7 each) In each of the following problems, either show that the statement is always true or provide a counterexample that shows it is false.

   (a) If K is a positive definite matrix, then K^2 is also positive definite.

   (b) The eigenvectors of square matrix A are also the eigenvectors of A^n for any positive integer n.

   (c) The set of complex vectors $\left\{ \begin{pmatrix} 1 \\ i \end{pmatrix}, \begin{pmatrix} 1+i \\ -1+i \end{pmatrix} \right\}$ spans $\mathbb{C}^2$.

Solution: (a) This is true. Since K is positive definite, it is symmetric and we have

$$K^2 = K^T K$$

so K^2 is a Gram matrix. Since K is positive definite, K is nonsingular so has linearly independent columns. The Gram matrix made is therefore positive definite, and

$$K^2 = K^T K > 0$$

(b) This is true. If $\mathbf{v}$ is an eigenvector of A then we have

$$A\mathbf{v} = \lambda \mathbf{v}$$

If we multiply $\mathbf{v}$ by A n times we have

$$\begin{aligned} A^n\mathbf{v} &= A^{n-1}A\mathbf{v} \\ &= A^{n-1}\lambda\mathbf{v} \\ &= A^{n-2}\lambda A\mathbf{v} \\ &= A^{n-2}\lambda^2\mathbf{v} \\ &= A^{n-3}\lambda^2 A\mathbf{v} \\ &= A^{n-3}\lambda^3\mathbf{v} \\ &\vdots \\ A^n\mathbf{v} &= \lambda^n\mathbf{v} \end{aligned}$$

which is the eigenvalue equation for A^n. So A^n has eigenvector $\mathbf{v}$ with eigenvalue λ^n.

(c) This is false as we can see by row reducing the matrix with the vectors as it's columns:

$$\begin{pmatrix} 1 & 1+i \\ i & -1+i \end{pmatrix} \xrightarrow[R_2 \to R_2 - iR_1]{} \begin{pmatrix} 1 & 1+i \\ 0 & 0 \end{pmatrix}$$

The second column is $1+i$ times the first column. Any $\mathbb{C}^2$ vector that isn't a multiple of the first column works as a counterexample, for example $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

2. (16 points) Let A = $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 1 \\ 1 & 2 & 1 \end{pmatrix}$ and let K be the Gram Matrixe given by $K = A^T A$.

   (a) (6 points) Show that $K = A^T A$ is not positive definite.

   (b) (8 points) Find all the null directions of K.

   (c) (2 points) What is the determinant of K?

Solution: (a) Since K is a Gram matrix equal to $A^T A$, it is only positive definite when the columns of A are linearly independent. We see by row reducing A that it's columns are linearly dependent:

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 1 \\ 1 & 2 & 1 \end{pmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

The matrix is singular, so we conclude that $A^T A$ must be only positive semidefinite, not positive definite.

(b) The null directions for K are all of the basis vectors for $\ker K$ which is the same as $\ker A$. Using the REF of A calculated above, we find that our null direction is

$$\mathbf{v} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

(c) Since A has linearly dependent columns, we know that $\det A = 0$. (This can also be seen from the REF.) This implies that $\det K = 0$ as well:

$$\begin{aligned} K &= A^T A \\ \det K &= \det(A^T A) \\ &= \det A^T \det A \\ &= (\det A)^2 \\ &= 0 \end{aligned}$$

3. (20 points) For the following questions, consider the vector space of real 2×2 matrices, $\mathbb{R}^{2\times 2}$, and use the inner product given by

$$\langle A, B \rangle = a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22}$$

   (a) (8 points) Show that $\langle A, B \rangle$ is a valid inner product in this vector space.

   (b) (5 points) Verify the Cauchy-Schwarz identity for the matrices

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad\qquad B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

   (c) (2 points) What is the angle between these two matrices?

   (d) (5 points) Verify the triangle inequality for these two matrices.

Solution: (a) This is a valid inner product:

i. It is clearly symmetric: $\langle A, B \rangle = \langle B, A \rangle$

ii. It has positivity: $\langle A, A \rangle = a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2 > 0$ unless $A = O_{2\times 2}$.

iii. It is bilinear:

$$\begin{aligned}\langle \alpha A + \gamma C, B \rangle &= (\alpha a_{11} + \gamma c_{11})b_{11} + (\alpha a_{12} + \gamma c_{12})b_{12} + (\alpha a_{21} + \gamma c_{21})b_{21} + (\alpha a_{22} + \gamma c_{22})b_{22} \\ &= \alpha a_{11}b_{11} + \alpha a_{12}b_{12} + \alpha a_{21}b_{21} + \alpha a_{22}b_{22} + \gamma c_{11}b_{11} + \gamma c_{12}b_{12} + \gamma c_{21}b_{21} + \gamma c_{22}b_{22} \\ &= \alpha \langle A, B \rangle + \gamma \langle C, B \rangle\end{aligned}$$

Symmetry then implies that

$$\langle A, \beta B + \gamma C \rangle = \beta \langle A, B \rangle + \gamma \langle A, C \rangle$$

(b) We calculate the norms of the matrices as well as their inner product:

$$\begin{aligned}||A||^2 &= \langle A, A \rangle \\ &= 1^2 + 0^2 + 0^2 + (-1)^2 = 2 \\ ||A|| &= \sqrt{2}\end{aligned}$$

$$\begin{aligned}||B||^2 &= \langle B, B \rangle \\ &= 1^2 + 1^2 + 1^2 + 1^2 = 4 \\ ||B|| &= 2\end{aligned}$$

$$\langle A, B \rangle = 1(1) + 0(1) + 0(1) + 1(-1) = 0$$

Cauchy-Schwarz says

$$|\langle A, B \rangle| \le ||A|| \cdot ||B||$$

Substituting the values above we have

$$0 \le 2\sqrt{2}$$

so C-S is verified.

(c) As $\langle A, B \rangle = 0$, the matrices are orthogonal, so the angle between them is $\frac{\pi}{2}$.

(d) To verify the triangle inequality we must find the norm of $A + B$:

$$\begin{aligned}||A+B||^2 &= \left|\left|\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}\right|\right|^2 \\ &= 2^2 + 1^2 + 1^2 + 0^2 = 6\end{aligned}$$

$$||A+B|| = \sqrt{6}$$

The triangle inequality states that

$$||A+B|| \leq ||A|| + ||B||$$

Substituting the above values gives

$$\sqrt{6} \leq \sqrt{2} + 2$$

squaring both sides and simplifying:

$$\begin{aligned}6 \leq& 2 + 4\sqrt{2} + 4 \\ 6 \leq& 6 + 4\sqrt{2} \\ 0 \leq& 4\sqrt{2}\end{aligned}$$

so the triangle inequality is verified.

4. (21 points) Matrix $A = \begin{pmatrix} 0 & 4 & 5 \\ 2 & 2 & 3 \\ 0 & 3 & 5 \end{pmatrix}$ has the QR factorization

$$A = \begin{pmatrix} 0 & 4/5 & -3/5 \\ 1 & 0 & 0 \\ 0 & 3/5 & 4/5 \end{pmatrix} \begin{pmatrix} 2 & 2 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & 1 \end{pmatrix}$$

   (a) (16 points) Use one of the techniques learned in class to calculate this or an equivalent QR factorization.

   (b) (5 points) Use this QR factorization to solve $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$

Solution: (a) This matrix can be factored into QR using either the Gram-Schmidt process or Householder matrices. In these solutions, we'll do it both ways.

Householder solution:

We want to find H_1 that reflects the first column of A, which has a norm of 2 into the vector $2\mathbf{e}_1$. Since we know that permutation matrices are themselves Householder matrices, we immediately see that

$$H_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and so after the first multiplication we have

$$\begin{aligned} H_1 A &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 4 & 5 \\ 2 & 2 & 3 \\ 0 & 3 & 5 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 3 & 5 \end{pmatrix} \end{aligned}$$

We now need to reflect vector equal to the second column of $H_1 A$ with the first component made a zero into the $\mathbf{e}_2$ direction. We find the norm of this vector:

$$\begin{aligned} \mathbf{v}_2 &= \begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix} \\ ||\mathbf{v}_2||^2 &= 25 \\ ||\mathbf{v}_2|| &= 5 \end{aligned}$$

The unit vector we need for our H_2 matrix is therefore

$$\mathbf{u}_2 = \frac{\mathbf{v}_2 - 5\mathbf{e}_2}{||\mathbf{v}_2 - 5\mathbf{e}_2||}$$

so we have

$$\mathbf{v}_2 - 5\mathbf{e}_2 = \begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix} - \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix}$$

$$||\mathbf{v}_2 - 5\mathbf{e}_2||^2 = 10$$

$$||\mathbf{v}_2 - 5\mathbf{e}_2|| = \sqrt{10}$$

$$\mathbf{u}_2 = \frac{1}{\sqrt{10}} \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix}$$

Now use our formula to find H_2:

$$\begin{aligned} H_2 &= I - 2\mathbf{u}_2\mathbf{u}_2^T \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - 2\frac{1}{(\sqrt{10})^2} \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix} \begin{pmatrix} 0 & -1 & 3 \end{pmatrix} \\ &= \frac{1}{5} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix} - \frac{1}{5} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & -3 & 9 \end{pmatrix} \\ &= \frac{1}{5} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 4 & 3 \\ 0 & 3 & -4 \end{pmatrix} \end{aligned}$$

Now we have:

$$H_2 H_1 A = \frac{1}{5} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 4 & 3 \\ 0 & 3 & -4 \end{pmatrix} \begin{pmatrix} 2 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 3 & 5 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & -1 \end{pmatrix} = R$$

This is our R matrix. Our Q matrix is $H_1 H_2$, since the H matrices are all orthogonal and symmetric:

$$\begin{aligned} H_2 H_1 A &= R \\ A &= H_1 H_2 R \end{aligned}$$

We find Q:

$$\begin{aligned} Q &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{1}{5} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 4 & 3 \\ 0 & 3 & -4 \end{pmatrix} \\ &= \frac{1}{5} \begin{pmatrix} 0 & 4 & 3 \\ 5 & 0 & 0 \\ 0 & 3 & -4 \end{pmatrix} \end{aligned}$$

so we have our QR factorization:

$$A = \frac{1}{5} \begin{pmatrix} 0 & 4 & 3 \\ 5 & 0 & 0 \\ 0 & 3 & -4 \end{pmatrix} \begin{pmatrix} 2 & 2 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & -1 \end{pmatrix}$$

This is a valid QR factorization. To get to the version in the problem statement, we multiply the last column of Q and the last row of R by -1.

Gram-Schmidt solution:

We start by taking our first orthonormal vector to be the normalized first column of A and r_{11} to be it's magnitude:

$$\begin{aligned} r_{11} &= 2 \\ \mathbf{q}_1 &= \frac{1}{2}\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \end{aligned}$$

We now find r_{12}, r_{22}, and $\mathbf{q}_2$:

$$\begin{aligned} r_{12} &= \langle \mathbf{q}_1, \mathbf{a}_2 \rangle = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix} = 2 \\ r_{22} &= \sqrt{||\mathbf{a}_2|| - r_{12}^2} = \sqrt{4^2 + 2^2 + 3^2 - 2^2} \\ &= \sqrt{16 + 4 + 9 - 4} = \sqrt{25} = 5 \\ \mathbf{q}_2 &= \frac{1}{r_{22}}(\mathbf{a}_2 - r_{12}\mathbf{q}_1) \\ &= \frac{1}{5}\left(\begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix} - 2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) \\ &= \frac{1}{5}\begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix} \end{aligned}$$

Finally for r_{13}, r_{23}, r_{33}, and $\mathbf{q}_3$:

$$\begin{aligned} r_{13} &= \langle \mathbf{q}_1, \mathbf{a}_3 \rangle = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 5 \\ 3 \\ 5 \end{pmatrix} = 3 \\ r_{23} &= \langle \mathbf{q}_2, \mathbf{a}_3 \rangle = \frac{1}{5}\begin{pmatrix} 4 & 0 & 3 \end{pmatrix} \begin{pmatrix} 5 \\ 3 \\ 5 \end{pmatrix} = 7 \\ r_{33} &= \sqrt{||\mathbf{a}_3|| - r_{13}^2 - r_{23}^2} = \sqrt{5^2 + 3^2 + 5^2 - 3^2 - 7^2} \\ &= \sqrt{25 + 9 + 25 - 9 - 49} = 1 \\ \mathbf{q}_3 &= \frac{1}{r_{33}}(\mathbf{a}_3 - r_{13}\mathbf{q}_1 - r_{23}\mathbf{q}_2) \\ &= \begin{pmatrix} 5 \\ 3 \\ 5 \end{pmatrix} - 3\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \frac{7}{5}\begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix} \\ &= \frac{1}{5}\begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix} \end{aligned}$$

Constructing Q and R from these values gives the original factorization:

$$
\begin{aligned}
QR &= \begin{pmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \mathbf{q}_3 \end{pmatrix} \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{pmatrix} \\
&= \begin{pmatrix} 0 & 4/5 & -3/5 \\ 1 & 0 & 0 \\ 0 & 3/5 & 4/5 \end{pmatrix} \begin{pmatrix} 2 & 2 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

(b) We solve $A\mathbf{x} = \mathbf{b}$ by substituting the QR factorization for A and using the orthogonality of Q to get

$$
\begin{aligned}
QR\mathbf{x} &= \mathbf{b} \\
R\mathbf{x} &= Q^T\mathbf{b}
\end{aligned}
$$

We may either invert R or solve the augmented matrix $(R|Q^T\mathbf{b})$. Here wee do the latter:

$$
\begin{aligned}
Q^T\mathbf{b} &= \begin{pmatrix} 0 & 1 & 0 \\ 4/5 & 0 & 3/5 \\ -3/5 & 0 & 4/5 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \\
&= \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}
\end{aligned}
$$

Now solve the augmented matrix:

$$
\begin{aligned}
\left(\begin{array}{ccc|c} 2 & 2 & 3 & 3 \\ 0 & 5 & 7 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right) &\to \left(\begin{array}{ccc|c} 2 & 2 & 0 & 0 \\ 0 & 5 & 0 & -5 \\ 0 & 0 & 1 & 1 \end{array}\right) \\
&\to \left(\begin{array}{ccc|c} 2 & 2 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right) \\
&\to \left(\begin{array}{ccc|c} 2 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right) \\
&\to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right)
\end{aligned}
$$

so our solution is

$$
\mathbf{x} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}
$$

5. (22 points) Let $A = \begin{pmatrix} 2 & 4 & -8 \\ 0 & 0 & -2 \\ 1 & 2 & -5 \end{pmatrix}$

   (a) (4 points) Show that the eigenvalues of A are 0, -1, and -2.

   (b) (12 points) For each eigenvalue of A, find the eigenvector.

   (c) (6 points) Show that $\ker A$ and coimg$A$ are orthogonal complements of $\mathbb{R}^3$.

Solution: (a) We first find the characteristic equation for our matrix:

$$\begin{aligned} 0 &= \det(A - \lambda I) \\ &= \det \begin{pmatrix} 2-\lambda & 4 & -8 \\ 0 & -\lambda & -2 \\ 1 & 2 & -5-\lambda \end{pmatrix} \\ &= -\lambda \det \begin{pmatrix} 2-\lambda & -8 \\ 1 & -5-\lambda \end{pmatrix} + 2 \det \begin{pmatrix} 2-\lambda & 4 \\ 1 & 2 \end{pmatrix} \\ &= -\lambda(-10 + 3\lambda + \lambda^2 + 8) + 2(4 - 2\lambda - 4) \\ &= 2\lambda - 3\lambda^2 - \lambda^3 - 4\lambda \\ &= -\lambda^3 - 3\lambda^2 - 2\lambda \\ &= -\lambda(\lambda^2 + 3\lambda + 2) \\ &= -\lambda(\lambda + 1)(\lambda + 2) \end{aligned}$$

The solutions are our eigenvectors, so we have $\lambda = 0$, -1, and -2.

(b) To find the eigenvectors, we find a basis for the kernel of $A - \lambda I$ for each value of λ:
$\lambda = 0$:

$$\begin{aligned} A &= \begin{pmatrix} 2 & 4 & -8 \\ 0 & 0 & -2 \\ 1 & 2 & -5 \end{pmatrix} \\ &\to \begin{pmatrix} 2 & 4 & -8 \\ 0 & 0 & -2 \\ 0 & 0 & -1 \end{pmatrix} \\ &\to \begin{pmatrix} 1 & 2 & -4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

so we have a free variable in the second column:

$$\mathbf{v}_0 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

$\lambda = -1$:

$$A - (-1)I = \begin{pmatrix} 3 & 4 & -8 \\ 0 & 1 & -2 \\ 1 & 2 & -4 \end{pmatrix}$$
$$\to \begin{pmatrix} 1 & 2 & -4 \\ 0 & 1 & -2 \\ 3 & 4 & -8 \end{pmatrix}$$
$$\to \begin{pmatrix} 1 & 2 & -4 \\ 0 & 1 & -2 \\ 0 & -2 & 4 \end{pmatrix}$$
$$\to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

so we have a free variable in the third column:

$$\mathbf{v}_{-1} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

$\lambda = -2$:

$$A - (-2)I = \begin{pmatrix} 4 & 4 & -8 \\ 0 & 2 & -2 \\ 1 & 2 & -3 \end{pmatrix}$$
$$\to \begin{pmatrix} 1 & 2 & -3 \\ 0 & 2 & -2 \\ 4 & 4 & -8 \end{pmatrix}$$
$$\to \begin{pmatrix} 1 & 2 & -3 \\ 0 & 2 & -2 \\ 0 & -4 & 4 \end{pmatrix}$$
$$\to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

so we have a free variable in the third column:

$$\mathbf{v}_{-2} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(c) The kernel of A is spanned by the eigenvector for $\lambda = 0$ since it is all vectors such that $A\mathbf{x} = \mathbf{0}$:

$$\ker A = \operatorname{span}\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \right\}$$

To verify this is orthogonal to the coimage, we need to find a basis for it. Fortunately, we have already found the REF of A above:

$$\begin{pmatrix} 2 & 4 & -8 \\ 0 & 0 & -2 \\ 1 & 2 & -5 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Taking the transposes of the non-zero columns of the REF gives us our basis vectors for the coimage:

$$\text{coimg}A = \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Each basis vector of coimg$A$ is orthogonal to the basis vector of $\ker A$:

$$\begin{pmatrix} -2 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} = 0$$
$$\begin{pmatrix} -2 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0$$

so they are orthogonal subspaces. To show they are orthogonal complements, we need to show that the three basis vectors span $\mathbb{R}^3$ by creating the matrix out of them and showing it is nonsingular:

$$\begin{aligned} \det \begin{pmatrix} -2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 4 & 1 \end{pmatrix} &= \det \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} \\ &= (-2)(2) - (1)(1) \\ &= -5 \end{aligned}$$

The matrix is nonsingular, so the vectors span $\mathbb{R}^3$, and the kernel and coimage are therefore orthogonal complements of $\mathbb{R}^3$.