Write your name below. This exam is worth 100 points and has 5 questions. On each problem, you must show all your work to receive credit on that problem. You are allowed to use one page of notes (one sided). You cannot collaborate on the exam or seek outside help, nor can you use the recorded lectures, a calculator, any computational software, or material you find online.

Name:

1. (21 points, 7 each) In each of the following problems, either show that the statement is always true or provide a counterexample that shows it is false.

   (a) If K is a positive definite matrix, then K^2 is also positive definite.

   (b) The eigenvectors of square matrix A are also the eigenvectors of A^n for any positive integer n.

   (c) The set of complex vectors $\left\{ \begin{pmatrix} 1 \\ i \end{pmatrix}, \begin{pmatrix} 1+i \\ -1+i \end{pmatrix} \right\}$ spans $\mathbb{C}^2$.

2. (16 points) Let A $= \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 1 \\ 1 & 2 & 1 \end{pmatrix}$ and let K be the Gram Matrixe given by $K = A^T A$.

   (a) (6 points) Show that $K = A^T A$ is not positive definite.

   (b) (8 points) Find all the null directions of K.

   (c) (2 points) What is the determinant of K?

3. (20 points) For the following questions, consider the vector space of real 2×2 matrices, $\mathbb{R}^{2\times 2}$, and use the inner product given by

$$\langle A, B \rangle = a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22}$$

   (a) (8 points) Show that $\langle A, B \rangle$ is a valid inner product in this vector space.

   (b) (5 points) Verify the Cauchy-Schwarz identity for the matrices

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

   (c) (2 points) What is the angle between these two matrices?

   (d) (5 points) Verify the triangle inequality for these two matrices.

4. (21 points) Matrix $A = \begin{pmatrix} 0 & 4 & 5 \\ 2 & 2 & 3 \\ 0 & 3 & 5 \end{pmatrix}$ has the QR factorization

$$A = \begin{pmatrix} 0 & 4/5 & -3/5 \\ 1 & 0 & 0 \\ 0 & 3/5 & 4/5 \end{pmatrix} \begin{pmatrix} 2 & 2 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & 1 \end{pmatrix}$$

(a) (16 points) Use one of the techniques learned in class to calculate this or an equivalent QR factorization.

(b) (5 points) Use this QR factorization to solve $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$

5. (22 points) Let $A = \begin{pmatrix} 2 & 4 & -8 \\ 0 & 0 & -2 \\ 1 & 2 & -5 \end{pmatrix}$

   (a) (4 points) Show that the eigenvalues of A are 0, -1, and -2.

   (b) (12 points) For each eigenvalue of A, find the eigenvector.

   (c) (6 points) Show that $\ker A$ and $\operatorname{coimg} A$ are orthogonal complements of $\mathbb{R}^3$.