Write your name below. This exam is worth 100 points. On each problem, you must show all your work to receive credit on that problem. You are allowed to use one page of notes (one sided). You cannot collaborate on the exam or seek outside help, nor can you use the recorded lectures, a calculator, any computational software, or material you find online.

Name:

1. (21 points, 7 each) In each of the following problems, either show that the statement is always true or provide a counterexample that shows it is false.

   (a) If A and B are both singular matrices of the same size, then AB is also singular.

   (b) If A and B are both skew-symmetric (or anti-symmetric) matrices of the same size, then AB is symmetric.

   (c) If A and B are both square matrices of the same size, then $\det(A+B) = \det(A) + \det(B)$.

Solution: (a) This is true. Since A and B are singular, their determinants are 0:

$$\det A = 0 \qquad \det B = 0$$

and so the determinant of their product is also 0. Using the properties of the determinant:

$$\begin{aligned}\det AB &= \det A \det B \\ &= 0\end{aligned}$$

So AB is also singular.

(b) This is false. One possible counterexample is

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

The product AB is not symmetric:

$$AB = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(c) This is false. One possible counterexample is

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Both A and B have determinant 0, but

$$\begin{aligned}\det(A+B) &= \det I_2 \\ &= 1\end{aligned}$$

2. (20 points) Let A $= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 10 \\ 1 & 3 & 4 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$

   (a) (10 points) If A is regular find its LU factorization. If A is not regular but is non-singular, find a permuted LU factorization.

   (b) (2 points) Find the determinant of A using your factorization from part (a).

   (c) (8 points) Solve the equation $A\mathbf{x} = \mathbf{b}$.

Solution: (a) We begin by making the first column upper triangular:

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 10 \\ 1 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \\ 0 & 1 & 1 \end{pmatrix}$$

The matrix is not regular, so we introduce the permutation that interchanges rows 2 and 3. This is enough to make U upper triangular, so we have the permuted LU factorization:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 10 \\ 1 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{pmatrix}$$

(b) We have done one row interchange in our factorization, and since $\det U$ is the product of its diagonals, we have

$$\begin{aligned} \det A &= (-1)^1 \det U \\ &= (-1)(1)(1)(4) \\ &= -4 \end{aligned}$$

(c) Since we have a permuted LU factorization we must multiply the equation by our permutation matrix before we solve:

$$\begin{aligned} A\mathbf{x} &= \mathbf{b} \\ PA\mathbf{x} &= P\mathbf{b} \\ LU\mathbf{x} &= P\mathbf{b} \end{aligned}$$

First we permute $\mathbf{b}$:

$$\begin{aligned} P\mathbf{b} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} \end{aligned}$$

Then we let $U\mathbf{x} = \mathbf{c}$ which gives $L\mathbf{c} = P\mathbf{b}$ which we solve:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 1 & 1 & 0 & 3 \\ 2 & 0 & 1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -4 \end{array}\right)$$

So we have $\mathbf{c} = \begin{pmatrix} 2 & 1 & -4 \end{pmatrix}^T$.

Now substitute this into $U\mathbf{x} = \mathbf{c}$ and solve:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 4 & -4 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 2 & 0 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

So our solution is

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

3. (9 points) Let $A = \begin{pmatrix} 2 & k \\ k & 6 \end{pmatrix}$ with $k \in \mathbb{R}$ and let $\mathbf{b} = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$.

   (a) (5 points) For what values of k is the system $A\mathbf{x} = \mathbf{b}$ guaranteed to have a unique solution?

   (b) (4 points) For each k value where $A\mathbf{x} = \mathbf{b}$ doesn't have a unique solution, what is the compatibility condition on $\mathbf{b}$?

Solution: (a) We make the augmented matrix and reduce:

$$\left(\begin{array}{cc|c} 2 & k & x \\ k & 6 & y \end{array}\right) \xrightarrow[R_2 \to R_2 - \frac{k}{2}R_1]{} \left(\begin{array}{cc|c} 2 & k & x \\ 0 & 6 - \frac{k^2}{2} & y - \frac{kx}{2} \end{array}\right)$$

To have a unique solution, we must have a non-zero entry in the bottom right of the reduced matrix, so we have

$$\begin{aligned} 6 - \frac{k^2}{2} &\neq 0 \\ 12 - k^2 &\neq 0 \\ k^2 &\neq 12 \\ k &\neq \pm 2\sqrt{3} \end{aligned}$$

So for any real value for k except these two the system will have unique solutions.

(b) The system won't have unique solutions when the bottom right entry in the reduced matrix equals 0. This happens when k equals one of the two values from part (a). For our compatibility conditions, we set the last entry in the vector equal to zero for the k value:

$k = 2\sqrt{3}$:

$$\begin{aligned} y - \frac{2\sqrt{3}x}{2} &= 0 \\ y &= \sqrt{3}x \end{aligned}$$

$k = -2\sqrt{3}$:

$$\begin{aligned} y + \frac{2\sqrt{3}x}{2} &= 0 \\ y &= -\sqrt{3}x \end{aligned}$$

4. (16 points) The following two questions are unrelated.

   (a) (8 points) Are the matrices $\left\{ \begin{pmatrix} 3 & 1 \\ 0 & -2 \end{pmatrix}, \begin{pmatrix} 4 & 1 \\ 0 & -3 \end{pmatrix}, \begin{pmatrix} -2 & -1 \\ 0 & 1 \end{pmatrix} \right\}$ linearly independent members of $\mathbb{R}^{2\times 2}$? Justify your answer.

   (b) (8 points) Are the polynomials $\{x^2+1, x^2-1, x\}$ a basis for $\mathcal{P}^{(2)}$, the vector space of all polynomials with degree ≤ 2? Justify your answer.

Solution: (a) We want to show that the only solution to

$$c_1 \begin{pmatrix} 3 & 1 \\ 0 & -2 \end{pmatrix} + c_2 \begin{pmatrix} 4 & 1 \\ 0 & -3 \end{pmatrix} + c_3 \begin{pmatrix} -2 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

has all $c_i = 0$. Equating the left hand side to the right hand side entry by entry gives a system of four linear equations. We find this system's coefficient matrix and solve the homogeneous equation:

$$A = \begin{pmatrix} 3 & 4 & -2 \\ 1 & 1 & -1 \\ 0 & 0 & 0 \\ -2 & -3 & 1 \end{pmatrix} \xrightarrow[R_3 \leftrightarrow R_4]{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 1 & -1 \\ 3 & 4 & -2 \\ -2 & -3 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow[R_3 \to R_3 + 2R_1]{R_2 \to R_2 - 3R_1} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow[R_3 \to R_3 + R_2]{} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Since row 3 is not a pivot column, the system has a free variable and so has infinitely many solutions where not all the $c_i = 0$. So the matrices are not linearly independent.

(b) For these polynomials to span $\mathcal{P}^{(2)}$, they must span the space and be linearly independent. We need to determine if a general member of $\mathcal{P}^{(2)}$ can be written as a unique linear combination of the polynomials. Setting our $\mathcal{P}^{(2)}$ member equal to the linear combination gives

$$\begin{aligned} ax^2 + bx + d &= c_1(x^2+1) + c_2(x^2-1) + c_3(x) \\ &= (c_1+c_2)x^2 + c_3(x) + (c_1 - c_2) \end{aligned}$$

Equating the coefficients of the powers on each side gives a system of three linear equations:

$$\begin{aligned} c_1 + c_2 &= a \\ c_3 &= b \\ c_1 - c_2 &= d \end{aligned}$$

Converting this to an augmented matrix and reducing gives

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 0 & 0 & 1 & b \\ 1 & -1 & 0 & d \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 1 & -1 & 0 & d \\ 0 & 0 & 1 & b \end{array}\right)$$
$$\to \left(\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 0 & -2 & 0 & d-a \\ 0 & 0 & 1 & b \end{array}\right)$$

The system has a unique solution for all members of $\mathcal{P}^{(2)}$. The polynomials are therefore linearly independent and span $\mathcal{P}^{(2)}$, so they are a basis.

5. (14 points) For the following problems, determine if the subsets are subspaces of the given vector spaces.

   (a) (7 points) Are the solutions to the differential equation $u' = -2xu$ a subspace of the vector space of differentiable real valued functions?

   (b) (7 points) Are the matrices of the form $\begin{pmatrix} a & b \\ b & ab \end{pmatrix}$ with $a, b \in \mathbb{R}$ a subspace of $\mathbb{R}^{2\times 2}$?

Solution: (a) This is a subspace. It is not empty, as it contains the 0 element because $u(x) = 0$ is a solution to the differential equation. Now we check for closure:

Scalar multiplication: If $u(x)$ is a solution, is $h = cu(x)$ a solution for real scalar c?

$$\begin{aligned} h' &= (cu)' \\ &= cu' \\ &= c(-2xu) \\ &= -2x(cu) \\ &= -2xh \end{aligned}$$

So h is also a solution and the set is closed under scalar multiplication.

Vector addition: If $u(x)$ and $v(x)$ is $w(x) = u(x) + v(x)$ also a solution?

$$\begin{aligned} w' &= (u+v)' \\ &= u' + v' \\ &= -2xu - 2xv \\ &= -2x(u+v) \\ &= -2xw \end{aligned}$$

So w is also a solution and the set is closed under vector multiplication.

Since the set is non-empty and closed under both operations, it is a subspace.

(b) This is not a subspace. While it contains the 0 matrix, it is not closed under either of the operations. For example,

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

is a member with $a = 1$ and $b = 1$ but $2A$ is not a member.

$$2A = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$$

We can identify $a = 2$ from the 1,1 entry and $b = 2$ from the 1,2 or 2,1 entry. For this matrix to be a member, the 2,2 entry must be $ab = 4$, but it is not. The subset is not closed under scalar multiplication and so is not a subspace.

6. (20 points) Let $A = \begin{pmatrix} 1 & 4 & -2 & -1 \\ 3 & 0 & 0 & -1 \\ -2 & -2 & 1 & 1 \end{pmatrix}$

   (a) (3 points) What is the rank of A?

   (b) (3 points) What is $\dim \operatorname{coker} A$?

   (c) (9 points) Find a basis for $\ker A$

   (d) (5 points) Find a basis for $\operatorname{coimg} A$

Solution: We'll need the REF of A to answer these questions:

$$\begin{pmatrix} 1 & 4 & -2 & -1 \\ 3 & 0 & 0 & -1 \\ -2 & -2 & 1 & 1 \end{pmatrix} \xrightarrow[R_3 \to R_3 + 2R_1]{R_2 \to R_2 - 3R_1} \begin{pmatrix} 1 & 4 & -2 & -1 \\ 0 & -12 & 6 & 2 \\ 0 & 6 & -3 & -1 \end{pmatrix}$$

$$\xrightarrow[R_3 \to R_3 + \frac{1}{2}R_2]{} \begin{pmatrix} 1 & 4 & -2 & -1 \\ 0 & -12 & 6 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(a) Since A has two pivot columns, $\operatorname{rank} A = 2$.

(b) Since A has three rows and is rank 2, the cokernel has dimension $3 - 2 = 1$

(c) We have two free variables, x_3 and x_4. We set $x_3 = 1$ and $x_4 = 0$ to find our first basis vector:
Row 2:

$$-12x_2 + 6 = 0$$
$$x_2 = \frac{1}{2}$$

Row 1:

$$x_1 + 4\left(\frac{1}{2}\right) - 2 = 0$$
$$x_1 = 0$$

So our first basis vector, after clearing the fraction, is

$$\begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix}$$

now set $x_3 = 0$ and $x_4 = 1$ to find our second basis vector:
Row 2:

$$-12x_2 + 2 = 0$$
$$x_2 = \frac{1}{6}$$

Row 1:

$$x_1 + 4\left(\frac{1}{6}\right) - 1 = 0$$
$$x_1 = \frac{1}{3}$$

So our second basis vector, again clearing the fractions, is

$$\begin{pmatrix} 2 \\ 1 \\ 0 \\ 6 \end{pmatrix}$$

And our basis is the set of these two vectors:

$$\left\{ \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \\ 6 \end{pmatrix} \right\}$$

(d) For the basis of the coimage, we take the non-zero rows of the REF and transpose them:

$$\left\{ \begin{pmatrix} 1 \\ 4 \\ -2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -12 \\ 6 \\ 2 \end{pmatrix} \right\}$$