Write your name below. This exam is worth 100 points. On each problem, you must show all your work to receive credit on that problem. You are allowed to use one page of notes (one sided). You cannot collaborate on the exam or seek outside help, nor can you use the recorded lectures, a calculator, any computational software, or material you find online.

Name:

1. (21 points, 7 each) In each of the following problems, either show that the statement is always true or provide a counterexample that shows it is false.

   (a) If A and B are both singular matrices of the same size, then AB is also singular.

   (b) If A and B are both skew-symmetric (or anti-symmetric) matrices of the same size, then AB is symmetric.

   (c) If A and B are both square matrices of the same size, then $\det(A+B) = \det(A) + \det(B)$.

2. (20 points) Let A $= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 10 \\ 1 & 3 & 4 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$

   (a) (10 points) If A is regular find its LU factorization. If A is not regular but is non-singular, find a permuted LU factorization.

   (b) (2 points) Find the determinant of A using your factorization from part (a).

   (c) (8 points) Solve the equation $A\mathbf{x} = \mathbf{b}$.

3. (9 points) Let $A = \begin{pmatrix} 2 & k \\ k & 6 \end{pmatrix}$ with $k \in \mathbb{R}$ and let $\mathbf{b} = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$.

   (a) (5 points) For what values of k is the system $A\mathbf{x} = \mathbf{b}$ guaranteed to have a unique solution?

   (b) (4 points) For each k value where $A\mathbf{x} = \mathbf{b}$ doesn't have a unique solution, what is the compatibility condition on $\mathbf{b}$?

4. (16 points) The following two questions are unrelated.

   (a) (8 points) Are the matrices $\left\{ \begin{pmatrix} 3 & 1 \\ 0 & -2 \end{pmatrix}, \begin{pmatrix} 4 & 1 \\ 0 & -3 \end{pmatrix}, \begin{pmatrix} -2 & -1 \\ 0 & 1 \end{pmatrix} \right\}$ linearly independent members of $\mathbb{R}^{2\times 2}$? Justify your answer.

   (b) (8 points) Are the polynomials $\left\{x^2+1, x^2-1, x\right\}$ a basis for $\mathcal{P}^{(2)}$, the vector space of all polynomials with degree ≤ 2? Justify your answer.

5. (14 points) For the following problems, determine if the subsets are subspaces of the given vector spaces.

   (a) (7 points) Are the solutions to the differential equation $u' = -2xu$ a subspace of the vector space of differentiable real valued functions?

   (b) (7 points) Are the matrices of the form $\begin{pmatrix} a & b \\ b & ab \end{pmatrix}$ with $a, b \in \mathbb{R}$ a subspace of $\mathbb{R}^{2\times 2}$?

6. (20 points) Let $A = \begin{pmatrix} 1 & 4 & -2 & -1 \\ 3 & 0 & 0 & -1 \\ -2 & -2 & 1 & 1 \end{pmatrix}$

   (a) (3 points) What is the rank of A?

   (b) (3 points) What is $\dim \operatorname{coker} A$?

   (c) (9 points) Find a basis for $\ker A$

   (d) (5 points) Find a basis for $\operatorname{coimg} A$