1. [2350/072525 (18 pts)] Let $\mathbf{F} = e^y\,\mathbf{i} + (xe^y + \sin z)\,\mathbf{j} + y\cos z\,\mathbf{k}$.

   (a) (6 pts) Show that $\mathbf{F}$ is conservative.

   (b) (12 pts) Find the work done by $\mathbf{F}$ on an object that moves from $(0,0,0)$ to $(1,-1,3)$.

SOLUTION:

(a)

$$\begin{aligned}\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^y & xe^y + \sin z & y\cos z \end{vmatrix} \\ &= \left[\frac{\partial}{\partial y}(y\cos z) - \frac{\partial}{\partial z}(xe^y + \sin z)\right]\mathbf{i} + \left[\frac{\partial}{\partial z}(e^y) - \frac{\partial}{\partial x}(y\cos z)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}(xe^y + \sin z) - \frac{\partial}{\partial y}(e^y)\right]\mathbf{k} \\ &= (\cos z - \cos z)\,\mathbf{i} + (0-0)\,\mathbf{j} + (e^y - e^y)\,\mathbf{k} = \mathbf{0}\end{aligned}$$

Since the vector field is defined on all of $\mathbb{R}^3$, which is simply connected, $\mathbf{F}$ is conservative.

(b) Since $\mathbf{F}$ is conservative a potential function, f, exists such that $\mathbf{F} = \nabla f$. Moreover, the work done by the force is independent of the path and is simply equal to the difference between the value of the potential function at the end of the path and that at the beginning. Find the potential function:

$$\frac{\partial f}{\partial x} = e^y \implies f(x,y,z) = \int e^y \,\mathrm{d}x = xe^y + g(y,z)$$

$$\frac{\partial f}{\partial y} = xe^y + g_y(y,z) = xe^y + \sin z \implies g_y(y,z) = \sin z \implies g(y,z) = \int \sin z \,\mathrm{d}y = y\sin z + h(z)$$

$$\implies f(x,y,z) = xe^y + y\sin z + h(z)$$

$$\frac{\partial f}{\partial z} = y\cos z + h'(z) = y\cos z \implies h'(z) = 0 \implies h(z) = \text{constant (which we can set to zero)}$$

giving the potential function as $f(x,y,z) = xe^y + y\sin z$. Using this we have

$$\text{Work} = \int_{\mathcal{C}} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_{(0,0,0)}^{(1,-1,3)} \nabla f \cdot \mathrm{d}\mathbf{r} = f(1,-1,3) - f(0,0,0) = e^{-1} - \sin 3$$

Remark: Since the vector field is conservative, you could compute a line integral using the line segment between the two points, but even with that simple path the line integral requires more work (no pun intended).

■

2. [2350/072525 (16 pts)] Consider the function $f(x,y) = x + xy - y$. For the following questions, you don't need to find any actual values, simply justify your answers.

   (a) (8 pts) Does $f(x,y)$ possess any local (relative) extreme values on $\mathbb{R}^2$?

   (b) (8 pts) Does $f(x,y)$ possess any extreme values (local/relative or global/absolute) on the region $|x-1| \le 1,\ |y+1| \le 1$?

SOLUTION:

(a)

$$f_x = 1 + y = 0 \implies y = -1 \quad f_y = x - 1 = 0 \implies x = 1 \implies (x,y) = (1,-1) \text{ is the sole critical point}$$

$$f_{xx} = f_{yy} = 0 \quad f_{xy} = 1 \implies D(x,y) = -1 < 0 \implies (1,-1) \text{ is a saddle point}$$

No local extreme values exist on $\mathbb{R}^2$.

(b) Since $f(x, y)$, a polynomial, is continuous everywhere, it is continuous on the region. The given region is a closed, bounded region and the Extreme Value Theorem applies, guaranteeing the existence of global extreme values.

■

3. [2350/072525 (20 pts)] Let $\mathcal{S}$ be the first octant portion of the plane with intercepts $(2, 0, 0)$, $(0, 4, 0)$ and $(0, 0, 1)$. Its surface area is $\sqrt{21}$. Using this information, find the average value of $f(x, y, z) = 1 + x$ on $\mathcal{S}$.

SOLUTION:

Since we know the intercepts of the plane, we can immediately write its equation as $\frac{x}{2} + \frac{y}{4} + z = 1$. (Note that one can also use the vectors $\langle 2, 0, -1\rangle$ and $\langle 2, -4, 0\rangle$ that lie in the plane to find its equation.) Since the surface is a function we can use the parameterization $\mathbf{r}(u, v) = \left\langle u, v, 1 - \frac{u}{2} - \frac{v}{4}\right\rangle$ with parameter domain $0 \le u \le 2, 0 \le v \le 4 - 2u$. This domain comes from the fact that the plane's first octant portion lies above the triangle with vertices $(0, 0)$, $(2, 0)$ and $(0, 4)$.

$$\mathbf{r}_u(u,v) = \left\langle 1, 0, -\frac{1}{2}\right\rangle, \quad \mathbf{r}_v(u,v) = \left\langle 0, 1, -\frac{1}{4}\right\rangle \implies \mathbf{r}_u \times \mathbf{r}_v = \left\langle \frac{1}{2}, \frac{1}{4}, 1\right\rangle \implies \|\mathbf{r}_u \times \mathbf{r}_v\| = \frac{\sqrt{21}}{4}$$

$$f(u, v) = 1 + u$$

$$\iint_{\mathcal{S}} (1 + x)\,\mathrm{d}S = \int_0^2 \int_0^{4-2u} (1 + u)\left(\frac{\sqrt{21}}{4}\right) \mathrm{d}v\,\mathrm{d}u = \frac{\sqrt{21}}{4}\int_0^2 (1 + u)(4 - 2u)\,\mathrm{d}u$$

$$= \frac{\sqrt{21}}{4}\int_0^2 \left(4 + 2u - 2u^2\right)\mathrm{d}u = \frac{\sqrt{21}}{4}\left(4u + u^2 - \frac{2u^3}{3}\right)\Big|_0^2 = \frac{\sqrt{21}}{4}\left(8 + 4 - \frac{16}{3}\right) = \frac{5\sqrt{21}}{3}$$

The average value we seek is thus $\frac{5\sqrt{21}}{3}/\sqrt{21} = \dfrac{5}{3}$.

■

4. [2350/072525 (18 pts)] Use Green's Theorem to find the outward flux of the vector field $\mathbf{F} = x^2y\,\mathbf{i} + 3xy^2\,\mathbf{j}$ through the boundary of the second quadrant portion of the circle of radius 3 centered at the origin. No points awarded if Green's Theorem is not used.

SOLUTION:

The boundary, $\partial\mathcal{D}$, consists of three piecewise smooth curves, two lines and a quarter circle surrounding the region $\mathcal{D}$. With $\mathbf{F} = \langle P, Q\rangle = \left\langle x^2y, 3xy^2\right\rangle$ we have

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 2xy + 6xy = 8xy$$

and using Green's Theorem gives

$$\text{Flux} = \int_{\partial\mathcal{D}} \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}s = \int_{\partial\mathcal{D}} x^2y\,\mathrm{d}y - 3xy^2\,\mathrm{d}x = \iint_{\mathcal{D}} 8xy\,\mathrm{d}A = 8\int_{-3}^{0}\int_0^{\sqrt{9-x^2}} xy\,\mathrm{d}A \quad \text{switch to polar coordinates}$$

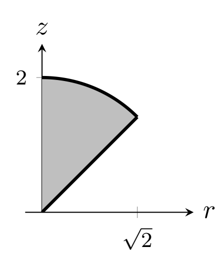
$$= 8\int_{\pi/2}^{\pi}\int_0^3 r^3\cos\theta\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta = 8\int_{\pi/2}^{\pi} \frac{r^4}{4}\Big|_0^3 \frac{1}{2}\sin 2\theta\,\mathrm{d}\theta = \frac{81}{2}\cos 2\theta\Big|_{\pi}^{\pi/2} = -81$$

Remark: The integration in rectangular coordinates isn't too bad, making the switch to polar coordinates somewhat optional. ■

5. [2350/072525 (20 pts)] Use Gauss' Divergence Theorem to find the outward flux of the vector field $\mathbf{F} = x^3\,\mathbf{i} + y^3\,\mathbf{j} + z^3\,\mathbf{k}$ through the boundary of the solid region above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 4$. No points awarded if Gauss' Divergence Theorem is not used.

SOLUTION:

Let $\mathcal{E}$ be the region over whose boundary $(\mathcal{S})$ we wish to compute the flux of $\mathbf{F}$. A sketch of the region in the rz-plane (constant θ):



$$\nabla \cdot \mathbf{F} = 3\left(x^2+y^2+z^2\right)$$

Using Gauss' Divergence Theorem we have

$$\begin{aligned}
\iint_{\mathcal{S}} \mathbf{F}\cdot \mathrm{d}\mathbf{S} &= \iiint_{\mathcal{E}} \nabla \cdot \mathbf{F}\,\mathrm{d}V \\
&= \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{4-x^2-y^2}} 3\left(x^2+y^2+z^2\right)\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}z \quad \text{rectangular coordinates} \\
&= \int_0^{2\pi} \int_0^{\sqrt{2}} \int_r^{\sqrt{4-r^2}} 3\left(r^2+z^2\right) r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta \quad \text{cylindrical coordinates} \\
&= \int_0^{2\pi} \int_0^{\pi/4} \int_0^{2} 3\left(\rho^2\right)\rho^2 \sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta \quad \text{spherical coordinates} \\
&= \frac{96\pi}{5}\left(2-\sqrt{2}\right)
\end{aligned}$$

■

6. [2350/072525 (18 pts)] A piece of wire is in the shape of (e^t, t^2) with its left end at the point $(1,0)$. The charge density on the wire is $q(x,y) = \dfrac{3y}{\sqrt{x^2+4y}}$ Coulombs per meter. If the total charge on the wire is 8 Coulombs, find the coordinates of the right end of the wire.

SOLUTION:

We need to compute the scalar line integral

$$\text{Total Charge} = \int_{\mathcal{C}} q(x,y)\,\mathrm{d}s$$

The left point of the wire occurs when $t = 0$ and we need to find the upper bound, say b, for t which we then can use to find the coordinates of the other end of the wire. The curve (wire) is

$$\begin{aligned}
\mathbf{r}(t) &= \left\langle e^t, t^2\right\rangle, \quad 0 \le t \le b \\
\mathbf{r}'(t) &= \left\langle e^t, 2t\right\rangle \\
\|\mathbf{r}'(t)\| &= \sqrt{e^{2t}+4t^2} \\
q[\mathbf{r}(t)] &= \frac{3t^2}{\sqrt{e^{2t}+4t^2}}
\end{aligned}$$

We then have

$$8 = \int_{\mathcal{C}} \frac{3y}{\sqrt{x^2+4y}}\,\mathrm{d}s = \int_0^b q[\mathbf{r}(t)]\,\|\mathbf{r}'(t)\|\,\mathrm{d}t = \int_0^b 3t^2\,\mathrm{d}t = b^3 \implies b = 2$$

giving the coordinates of the right end of the wire as $\mathbf{r}(2) = (e^2, 4)$.

■

7. [2350/072525 (40 pts)] Let $\mathbf{V} = y\,\mathbf{i} + yz\,\mathbf{j} - \frac{1}{2}x^2\,\mathbf{k}$ be the velocity of a fluid and consider the surface, $\mathcal{S}$, given by $x^2+y^2-z^2 = -1$, $1 \le z \le \sqrt{5}$ with upward pointing normal.

   (a) (5 pts) Name the surface.

   (b) (15 pts) Find the circulation of $\mathbf{V}$ on the boundary of $\mathcal{S}$ by direct computation. The identity $1 - 2\sin^2 t = \cos 2t$ may be helpful.

   (c) (20 pts) Find the circulation of $\mathbf{V}$ on the boundary of $\mathcal{S}$ using Stokes' Theorem. No points awarded if Stokes' Theorem not used.

SOLUTION:

(a) The surface is the upper branch of a hyperboloid of two sheets.

(b) The boundary of the surface, $\partial\mathcal{S}$, is obtained by setting $z = \sqrt{5}$, that is, $x^2 + y^2 = 4$. The upward pointing normal induces a counterclockwise (when viewed from above) orientation on the boundary. Parameterizing this boundary gives

$$\mathbf{r}(t) = \langle 2\cos t, 2\sin t, \sqrt{5}\rangle \quad 0 \le t \le 2\pi$$

$$\mathbf{r}'(t) = \langle -2\sin t, 2\cos t, 0\rangle$$

$$\mathbf{V}[\mathbf{r}(t)] = 2\sin t\,\mathbf{i} + 2\sqrt{5}\sin t\,\mathbf{j} - 2\cos^2 t\,\mathbf{k}$$

$$\mathbf{V}[\mathbf{r}(t)]\cdot\mathbf{r}'(t) = \langle 2\sin t, 2\sqrt{5}\sin t, -2\cos^2 t\rangle \cdot \langle -2\sin t, 2\cos t, 0\rangle = -4\sin^2 t + 4\sqrt{5}\sin t\cos t$$

$$\text{Circulation} = \int_{\partial\mathcal{S}} \mathbf{V}\cdot \mathrm{d}\mathbf{r} = \int_0^{2\pi} \left(-4\sin^2 t + 4\sqrt{5}\sin t\cos t\right)\mathrm{d}t$$

$$= 2\int_0^{2\pi} \left(\cos 2t - 1 + \sqrt{5}\sin 2t\right)\mathrm{d}t = -4\pi$$

(c) We need the curl of the vector field

$$\boldsymbol{\nabla}\times\mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & yz & -\frac{1}{2}x^2 \end{vmatrix} = \langle -y, x, -1\rangle$$

We can parameterize the surface (hyperboloid) in two natural ways.

Method 1:

$$\mathbf{r}(u,v) = \left\langle u, v, \sqrt{1+u^2+v^2}\right\rangle,\ 0 \le u^2+v^2 \le 4$$

$$\mathbf{r}_u = \left\langle 1, 0, \frac{u}{\sqrt{1+u^2+v^2}}\right\rangle, \quad \mathbf{r}_v = \left\langle 0, 1, \frac{v}{\sqrt{1+u^2+v^2}}\right\rangle, \quad \mathbf{r}_u\times\mathbf{r}_v = \left\langle \frac{-u}{\sqrt{1+u^2+v^2}}, \frac{-v}{\sqrt{1+u^2+v^2}}, 1\right\rangle$$

This has the correct orientation. Then

$$\boldsymbol{\nabla}\times\mathbf{V}[\mathbf{r}(u,v)]\cdot(\mathbf{r}_u\times\mathbf{r}_v) = \langle -v, u, -1\rangle \cdot \left\langle \frac{-u}{\sqrt{1+u^2+v^2}}, \frac{-v}{\sqrt{1+u^2+v^2}}, 1\right\rangle = -1$$

$$\text{Circulation} = \int_{\partial\mathcal{S}} \mathbf{V}\cdot\mathrm{d}\mathbf{r} = \iint_{\mathcal{S}} \boldsymbol{\nabla}\times\mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S = \iint\limits_{u^2+v^2\le 4} -1\,\mathrm{d}A = -4\pi$$

Method 2:

$$\mathbf{r}(u,v) = \left\langle v\cos u, v\sin u, \sqrt{1+v^2}\right\rangle,\ 0 \le u \le 2\pi,\ 0 \le v \le 2$$

$$\mathbf{r}_u = \langle -v\sin u, v\cos u, 0\rangle, \quad \mathbf{r}_v = \left\langle \cos u, \sin u, \frac{v}{\sqrt{1+v^2}}\right\rangle, \quad \mathbf{r}_u\times\mathbf{r}_v = \left\langle \frac{v^2\cos u}{\sqrt{1+v^2}}, \frac{v^2\sin u}{\sqrt{1+v^2}}, -v\right\rangle$$

This has the incorrect orientation so we'll use $-\mathbf{r}_u\times\mathbf{r}_v$. Then

$$\boldsymbol{\nabla}\times\mathbf{V}[\mathbf{r}(u,v)]\cdot(-\mathbf{r}_u\times\mathbf{r}_v) = \langle -v\sin u, v\cos u, -1\rangle \cdot \left\langle \frac{-v^2\cos u}{\sqrt{1+v^2}}, \frac{-v^2\sin u}{\sqrt{1+v^2}}, v\right\rangle = -v$$

$$\text{Circulation} = \int_{\partial\mathcal{S}} \mathbf{V}\cdot\mathrm{d}\mathbf{r} = \iint_{\mathcal{S}} \boldsymbol{\nabla}\times\mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S = \int_0^{2\pi}\int_0^2 -v\,\mathrm{d}v\,\mathrm{d}u = -4\pi$$

Note that the horizontal "cap" on the hyperboloid (a disk of radius 2 in the plane $z = \sqrt{5}$) shares the boundary with the hyperboloid itself.. Therefore, this can be used as the surface in Stokes' Theorem. It, too, can be parameterized two ways:

$$\mathbf{r}(u,v) = \langle u, v, \sqrt{5}\rangle, \quad u^2+v^2 \le 4 \text{ or}$$

$$\mathbf{r}(u,v) = \langle v\cos u, v\sin u, \sqrt{5}\rangle, \quad 0 \le u \le 2\pi,\ 0 \le v \le 2$$

■