1. [2360/072525 (10 pts)] Let $L(y)$ represent a linear operator describing a fourth order differential equation. Consider the set of solutions to $L(y) = 0$ given by $\{t-1, 2t+1, t^2-7t+3, 4t^2+8t\}$. Does this set constitute a basis for the solution space of $L(y) = 0$? Justify your answer completely.

**SOLUTION:**

The dimension of the solution space is four so if this set is linearly independent, then it will form a basis.

$$W[t-1, 2t+1, t^2-7t+3, 4t^2+8t](t) = \begin{vmatrix} t-1 & 2t+1 & t^2-7t+3 & 4t^2+8t \\ 1 & 2 & 2t-7 & 8t+8 \\ 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0 \quad \text{(note row of zeros)}$$

Since the Wronksian vanishes identically and the functions are solutions to a differential equation, they are linearly dependent and thus cannot form a basis for the solution space.

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2. [2360/072525 (12 pts)] Solve the IVP $\quad (4-t)\dfrac{\mathrm{d}x}{\mathrm{d}t} = 20 - 5t - 2x,\ x(0) = 4$. Give the largest interval over which the solution is valid.

**SOLUTION:**

The problem can be solved either using Euler-Lagrange two-step method (variation of parameters) or integrating factor, show here.

$$x' + \frac{2}{4-t}x = 5$$

$$\int \frac{2}{4-t}\,\mathrm{d}t = -2\ln|4-t| = \ln|4-t|^{-2} \implies \mu(t) = (4-t)^{-2}$$

$$\left[(4-t)^{-2}x\right]' = 5(4-t)^{-2}$$

$$(4-t)^{-2}x(t) = 5(4-t)^{-1} + C$$

$$x(t) = 5(4-t) + c(4-t)^2 \qquad \text{apply initial condition}$$

$$x(0) = 4 = 20 + c(16) \implies c = -1$$

$$x(t) = 5(4-t) - 1(4-t)^2 = 20 - 5t - (16 - 8t + t^2) = 4 + 3t - t^2 \qquad \text{valid on } (-\infty, 4)$$

■

3. [2360/072525 (16 pts)] A critically damped, unforced harmonic oscillator consisting of a one-quarter kilogram mass and a spring with a restoring constant of 25 newtons per meter is oriented horizontally. Let $x(t)$ be the position of the mass at time $t$. Use the standard sign conventions for displacement and velocity.

   (a) (4 pts) Write the differential equation governing the motion of the oscillator.

   (b) (2 pts) Suppose the motion is started when $t = 0$ by pushing the mass to the left at 10 meters per second from a position 7 meters to the right of the rest position. What are the initial conditions?

   (c) (2 pts) If the oscillator were undamped and driven by the function $f(t) = \cos\omega_f t$, what value of $\omega_f$ would result in unbounded solutions to the initial value problem?

   (d) (8 pts) Now suppose an external driving force, $f(t)$, is applied to the oscillator as follows: There is no driving force for the first 5 seconds. At 5 seconds, a driving force of $t-5$ is applied. Five seconds later the driving force is $5e^{-(t-10)}$. Finally, at 20 seconds, a unit impulse is applied. Write this driving force as a single function (not piecewise).

**SOLUTION:**

   (a) We have $m = 1/4, k = 25$ and since the oscillator is critically damped, $b^2 - 4\left(\frac{1}{4}\right)(25) = 0 \implies b = 5$. The differential equation is thus

$$\frac{1}{4}\ddot{x} + 5\dot{x} + 25x = 0$$

   (b) $x(0) = 7, \dot{x}(0) = -10$

(c) Unbounded solutions are the result of the oscillator being in resonance.

$$\omega_f = \sqrt{\frac{25}{1/4}} = 10$$

(d) $f(t) = (t-5)\text{step}(t-5) - (t-5)\text{step}(t-10) + 5e^{-(t-10)}\text{step}(t-10) + \delta(t-20)$

■

4. [2360/072525 (33 pts)] Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}$.

(a) (5 pts) One of the eigenvalues of $\mathbf{A}$ is $\lambda = 1$. Calculate $\mathbf{A}\vec{\mathbf{x}}$ where $\vec{\mathbf{x}} = \begin{bmatrix} -1 & 4 & 1 \end{bmatrix}^{\mathrm{T}}$. What can you say about $\vec{\mathbf{x}}$?

(b) (10 pts) Another eigenvalue of $\mathbf{A}$ is $\lambda = 3$. Suppose after some elementary row operations on the augmented matrix associated with the linear system $(\mathbf{A} - 3\mathbf{I})\,\vec{\mathbf{v}} = \vec{\mathbf{0}}$ you obtain

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 5 & 0 & -5 & 0 \\ 2 & 1 & -4 & 0 \end{array}\right]$$

i. (5 pts) Put this matrix into RREF.

ii. (5 pts) Find a basis for the eigenspace associated with $\lambda = 3$. What is its dimension?

(c) (8 pts) The third eigenvalue of $\mathbf{A}$ is $\lambda = -2$ with associated eigenvector $\vec{\mathbf{v}} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$. Write the general solution of the system of differential equations $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}}$.

(d) (10 pts) Do the columns of $\mathbf{A}$ form a basis for $\mathbb{R}^3$? Justify your answer.

**SOLUTION:**

(a)

$$\mathbf{A}\vec{\mathbf{x}} = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} = 1\vec{\mathbf{x}} \implies \begin{bmatrix} -1 & 4 & 1 \end{bmatrix}^{\mathrm{T}} \quad \text{is the eigenvector associated with the eigenvalue 1}$$

(b) i.

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 5 & 0 & -5 & 0 \\ 2 & 1 & -4 & 0 \end{array}\right] \xrightarrow{R_2^* = \frac{1}{5}R_2} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 2 & 1 & -4 & 0 \end{array}\right] \xrightarrow{R_3^* = -2R_2 + R_3} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2; R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

ii. From the RREF, $v_1 = v_3, v_2 = 2v_3$ with $v_3$ the free variable. A basis for the eigenspace is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$ which has a dimension of 1.

(c)

$$\vec{\mathbf{x}}(t) = c_1 e^{t} \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + c_3 e^{-2t} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

(d) There are three vectors given and the dimension of $\mathbb{R}^3$ is three so we only need to check that the vectors are linearly independent. To this end, we need to show that the only solution of

$$c_1 \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 4 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

is $c_1 = c_2 = c_3 = 0$. This is equivalent to the linear system

$$\begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

with

$$\begin{vmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{vmatrix} = -6$$

showing that the system has only the trivial solution $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. This proves that the vectors are linearly independent implying that the columns of $\mathbf{A}$ do form a basis for $\mathbb{R}^3$.

■

5. [2360/072525 (20 pts)] Use Laplace transforms to solve the initial value problem

$$y'' + 25y = 25[1 - \text{step}(t-4)],\ y(0) = y'(0) = 0$$

**SOLUTION:**

Taking Laplace transforms of both sides yields

$$s^2Y(s) - sy(0) - y'(0) + 25Y(s) = 25\left(\frac{1-e^{-4s}}{s}\right)$$

$$Y(s) = 25\left(\frac{1-e^{-4s}}{s(s^2+25)}\right)$$

$$\frac{25}{s(s^2+25)} = \frac{A}{s} + \frac{Bs+C}{s^2+25} \text{ after some algebra} = \frac{1}{s} - \frac{s}{s^2+25}$$

$$Y(s) = \frac{1}{s} - \frac{s}{s^2+25} - e^{-4s}\left(\frac{1}{s} - \frac{s}{s^2+25}\right)$$

$$y(t) = \mathscr{L}^{-1}\{Y(s)\} = 1 - \cos 5t - [1 - \cos 5(t-4)]\,\text{step}(t-4)$$

■

6. [2360/072525 (20 pts)] Solve the initial value problem $\vec{\mathbf{x}}' = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \vec{\mathbf{x}},\ \vec{\mathbf{x}}(0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ knowing that one eigenvalue/eigenvector pair is $\lambda = 1 + i, \vec{\mathbf{v}} = \begin{bmatrix} i \\ 1 \end{bmatrix}$. Write your final answer as a single vector. Answers containing the imaginary unit $i = \sqrt{-1}$ will receive a maximum of 5 points.

**SOLUTION:**

We have $\alpha = 1, \beta = 1, \mathbf{p} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{q} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Thus the general solution is

$$\begin{aligned} \vec{\mathbf{x}}(t) &= c_1\left(e^t \cos t \begin{bmatrix} 0 \\ 1 \end{bmatrix} - e^t \sin t \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + c_2\left(e^t \sin t \begin{bmatrix} 0 \\ 1 \end{bmatrix} + e^t \cos t \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \\ &= \begin{bmatrix} -c_1 e^t \sin t + c_2 e^t \cos t \\ c_1 e^t \cos t + c_2 e^t \sin t \end{bmatrix} \end{aligned}$$

Applying the initial conditions yields

$$\begin{aligned} 0 + c_2 &= -1 \\ c_1 + 0 &= 1 \end{aligned}$$

giving the unique solution to the system of differential equations as

$$\vec{\mathbf{x}}(t) = e^t \begin{bmatrix} -\sin t - \cos t \\ \cos t - \sin t \end{bmatrix}$$

■

7. [2360/072525 (39 pts)] Make a legible table on your paper corresponding to the questions below and write the word **TRUE** or **FALSE** in the appropriate place in your table. Keep your answers separate from any scratch work to facilitate grading. No partial credit will be awarded and no work is required to be shown.

(a) Consider the linear system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ where $\mathbf{A}$ is an $n \times n$ matrix.

i. If $|\mathbf{A}| = 0$, and $\vec{\mathbf{b}} \neq \vec{\mathbf{0}}$ the linear system always has infinitely many solutions.

ii. If $\mathbf{A}$ has zero as an eigenvalue, the linear system with $\vec{\mathbf{b}} = \vec{\mathbf{0}}$ is consistent.

iii. The solution to the linear system is always $\vec{\mathbf{x}} = \mathbf{A}^{-1}\vec{\mathbf{b}}$.

iv. The solution space to the system where $\vec{\mathbf{b}}$ is an $n \times 1$ column vector of ones is a subspace of $\mathbb{R}^n$.

(b) Consider the system of differential equations $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}}$ where $\mathbf{A}$ is a $2 \times 2$ matrix.

i. If Tr $\mathbf{A} = 0$ and $|\mathbf{A}| \neq 0$, the fixed point at $(0, 0)$ is a stable node.

ii. If $(\text{Tr}\,\mathbf{A})^2 - 4|\mathbf{A}| < 0$ and Tr $\mathbf{A} < 0$, then all solutions of the system will approach 0 as $t \to \infty$.

iii. If $|\mathbf{A}| \neq 0$, the fixed point at $(0, 0)$ in the phase plane is always stable.

iv. The $v$ and $h$ nullclines are lines.

v. If Tr $\mathbf{A} = 10$ and $|\mathbf{A}| = 25$, the equilibrium solution must be an unstable degenerate node.

(c) Consider the differential equation $y' = 3t^2y^2$.

i. Picard's Theorem guarantees the existence of a unique solution for any initial condition $y(t_0) = y_0$.

ii. The unique solution to the initial value problem consisting of the differential equation and the initial condition $y(0) = 1$ is $y = -(t^3 - 1)^{-1}$.

iii. The equation is a second order linear homogeneous equation.

iv. Euler's method cannot be used to approximate solutions to the differential equation that pass through the origin.

**SOLUTION:**

(a) i. **FALSE** - the system may be inconsistent, in which case there are no solutions

ii. **TRUE** - homogeneous linear systems are always consistent

iii. **FALSE** - the inverse matrix may not exist

iv. **FALSE** - the system is nonhomogeneous and the solution set is not even a vector space much less a subspace since it does not contain the zero vector

(b) i. **FALSE** - it is either a center or saddle

ii. **TRUE** - this puts us above the parabola in the second quadrant

iii. **FALSE** - for example, if $|\mathbf{A}| < 0$ fixed point is a saddle which is unstable

iv. **TRUE** - the system has the form $x_1' = ax_1 + bx_2$ and $x_2' = cx_1 + dx_2$. Setting these to 0 results in lines.

v. **FALSE** - $(\text{Tr}\mathbf{A})^2 - 4|\mathbf{A}| = 0, \text{Tr}\mathbf{A} > 0$; equilibrium solution could be an unstable degenerate node **OR** star node, depending on the geometric multiplicity of the eigenvalue

(c) i. **TRUE** - $f(t, y) = t^2y^2$ and $f_y(t, y) = 2t^2y$ are continuous for all $t$ and $y$

ii. **TRUE** - plug the solution into the differential equation and see that an identity results; check that the initial condition is satisfied

iii. **FALSE** - it is first order and nonlinear

iv. **FALSE** - Since $f(t, y) = 3t^2y^2$ is defined for all $t$ and $y$, Euler's method is applicable regardless of the initial condition.

■