- This exam is worth 150 points and has 7 problems.
- Show all work and simplify your answers! Answers with no justification will receive no points unless otherwise noted.
- Begin each problem on a new page.
- **DO NOT LEAVE THE EXAM UNTIL YOUR HAVE SATISFACTORILY SCANNED AND UPLOADED YOUR EXAM TO GRADESCOPE.**
- You are taking this exam in a proctored and honor code enforced environment. No calculators, cell phones, or other electronic devices or the internet are permitted during the exam. You are allowed one 8.5"× 11" crib sheet with writing on two sides.
- Remote students are allowed use of a computer during the exam only for a live video of their hands and face and to view the exam in the Zoom meeting. Remote students cannot interact with anyone except the proctor during the exam.

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0. At the top of the page containing your solution to problem 1, write the following statement and sign your name to it: "On my honor, I confirm the work herein is mine and has not been created using any computer resources." **FAILURE TO INCLUDE THIS STATEMENT AND YOUR SIGNATURE MAY RESULT IN A PENALTY.**

1. [2360/072525 (10 pts)] Let $L(y)$ represent a linear operator describing a fourth order differential equation. Consider the set of solutions to $L(y) = 0$ given by $\{t-1, 2t+1, t^2-7t+3, 4t^2+8t\}$. Does this set constitute a basis for the solution space of $L(y) = 0$? Justify your answer completely.

2. [2360/072525 (12 pts)] Solve the IVP $(4-t)\dfrac{\mathrm{d}x}{\mathrm{d}t} = 20 - 5t - 2x,\ x(0) = 4$. Give the largest interval over which the solution is valid.

3. [2360/072525 (16 pts)] A critically damped, unforced harmonic oscillator consisting of a one-quarter kilogram mass and a spring with a restoring constant of 25 newtons per meter is oriented horizontally. Let $x(t)$ be the position of the mass at time t. Use the standard sign conventions for displacement and velocity.

   (a) (4 pts) Write the differential equation governing the motion of the oscillator.

   (b) (2 pts) Suppose the motion is started when $t = 0$ by pushing the mass to the left at 10 meters per second from a position 7 meters to the right of the rest position. What are the initial conditions?

   (c) (2 pts) If the oscillator were undamped and driven by the function $f(t) = \cos\omega_f t$, what value of ω_f would result in unbounded solutions to the initial value problem?

   (d) (8 pts) Now suppose an external driving force, $f(t)$, is applied to the oscillator as follows: There is no driving force for the first 5 seconds. At 5 seconds, a driving force of $t-5$ is applied. Five seconds later the driving force is $5e^{-(t-10)}$. Finally, at 20 seconds, a unit impulse is applied. Write this driving force as a single function (not piecewise).

4. [2360/072525 (33 pts)] Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}$.

   (a) (5 pts) One of the eigenvalues of $\mathbf{A}$ is $\lambda = 1$. Calculate $\mathbf{A}\vec{\mathbf{x}}$ where $\vec{\mathbf{x}} = \begin{bmatrix} -1 & 4 & 1 \end{bmatrix}^{\mathrm{T}}$. What can you say about $\vec{\mathbf{x}}$?

   (b) (10 pts) Another eigenvalue of $\mathbf{A}$ is $\lambda = 3$. Suppose after some elementary row operations on the augmented matrix associated with the linear system $(\mathbf{A} - 3\mathbf{I})\,\vec{\mathbf{v}} = \vec{\mathbf{0}}$ you obtain

   $$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 5 & 0 & -5 & 0 \\ 2 & 1 & -4 & 0 \end{array}\right]$$

      i. (5 pts) Put this matrix into RREF.

      ii. (5 pts) Find a basis for the eigenspace associated with $\lambda = 3$. What is its dimension?

   (c) (8 pts) The third eigenvalue of $\mathbf{A}$ is $\lambda = -2$ with associated eigenvector $\vec{\mathbf{v}} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$. Write the general solution of the system of differential equations $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}}$.

   (d) (10 pts) Do the columns of $\mathbf{A}$ form a basis for $\mathbb{R}^3$? Justify your answer.

MORE PROBLEMS AND LAPLACE TRANSFORM TABLE BELOW/ON REVERSE

5. [2360/072525 (20 pts)] Use Laplace transforms to solve the initial value problem

$$y'' + 25y = 25[1 - \text{step}(t-4)], \; y(0) = y'(0) = 0$$

6. [2360/072525 (20 pts)] Solve the initial value problem $\vec{\mathbf{x}}' = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \vec{\mathbf{x}}, \; \vec{\mathbf{x}}(0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ knowing that one eigenvalue/eigenvector pair is $\lambda = 1 + i, \vec{\mathbf{v}} = \begin{bmatrix} i \\ 1 \end{bmatrix}$. Write your final answer as a single vector. Answers containing the imaginary unit $i = \sqrt{-1}$ will receive a maximum of 5 points.

7. [2360/072525 (39 pts)] Make a legible table on your paper corresponding to the questions below and write the word **TRUE** or **FALSE** in the appropriate place in your table. Keep your answers separate from any scratch work to facilitate grading. No partial credit will be awarded and no work is required to be shown.

   (a) Consider the linear system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ where $\mathbf{A}$ is an $n \times n$ matrix.

      i. If $|\mathbf{A}| = 0$, and $\vec{\mathbf{b}} \neq \vec{\mathbf{0}}$ the linear system always has infinitely many solutions.

      ii. If $\mathbf{A}$ has zero as an eigenvalue, the linear system with $\vec{\mathbf{b}} = \vec{\mathbf{0}}$ is consistent.

      iii. The solution to the linear system is always $\vec{\mathbf{x}} = \mathbf{A}^{-1}\vec{\mathbf{b}}$.

      iv. The solution space to the system where $\vec{\mathbf{b}}$ is an $n \times 1$ column vector of ones is a subspace of $\mathbb{R}^n$.

   (b) Consider the system of differential equations $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}}$ where $\mathbf{A}$ is a 2×2 matrix.

      i. If Tr $\mathbf{A} = 0$ and $|\mathbf{A}| \neq 0$, the fixed point at $(0, 0)$ is a stable node.

      ii. If (Tr $\mathbf{A})^2 - 4|\mathbf{A}| < 0$ and Tr $\mathbf{A} < 0$, then all solutions of the system will approach 0 as $t \to \infty$.

      iii. If $|\mathbf{A}| \neq 0$, the fixed point at $(0, 0)$ in the phase plane is always stable.

      iv. The v and h nullclines are lines.

      v. If Tr $\mathbf{A} = 10$ and $|\mathbf{A}| = 25$, the equilibrium solution must be an unstable degenerate node.

   (c) Consider the differential equation $y' = 3t^2y^2$.

      i. Picard's Theorem guarantees the existence of a unique solution for any initial condition $y(t_0) = y_0$.

      ii. The unique solution to the initial value problem consisting of the differential equation and the initial condition $y(0) = 1$ is $y = -(t^3 - 1)^{-1}$.

      iii. The equation is a second order linear homogeneous equation.

      iv. Euler's method cannot be used to approximate solutions to the differential equation that pass through the origin.

Short table of Laplace Transforms: $\mathscr{L}\{f(t)\} = F(s) \equiv \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$

In this table, a, b, c are real numbers with $c \geq 0$, and $n = 0, 1, 2, 3, \ldots$

$$\mathscr{L}\left\{t^n e^{at}\right\} = \frac{n!}{(s-a)^{n+1}} \qquad \mathscr{L}\left\{e^{at} \cos bt\right\} = \frac{s-a}{(s-a)^2 + b^2} \qquad \mathscr{L}\left\{e^{at} \sin bt\right\} = \frac{b}{(s-a)^2 + b^2}$$

$$\mathscr{L}\{\cosh bt\} = \frac{s}{s^2 - b^2} \qquad \mathscr{L}\{\sinh bt\} = \frac{b}{s^2 - b^2}$$

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{\mathrm{d}^n F(s)}{\mathrm{d}s^n} \qquad \mathscr{L}\left\{e^{at} f(t)\right\} = F(s-a) \qquad \mathscr{L}\{\delta(t-c)\} = e^{-cs}$$

$$\mathscr{L}\{t f'(t)\} = -F(s) - s\frac{\mathrm{d}F(s)}{\mathrm{d}s} \qquad \mathscr{L}\{f(t-c)\,\text{step}(t-c)\} = e^{-cs}F(s) \qquad \mathscr{L}\{f(t)\,\text{step}(t-c)\} = e^{-cs}\mathscr{L}\{f(t+c)\}$$

$$\mathscr{L}\left\{f^{(n)}(t)\right\} = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \cdots - f^{(n-1)}(0)$$
