**Answer the following problems, showing all of your work and simplifying your solutions where possible unless otherwise stated.**

**No calculators, notes, books, electronic devices, internet access, AI tools etc. are allowed. This is a closed book, closed note exam.**

1. (30 pts) Are the following series convergent or divergent? Justify your answer and state which test was used.

   (a) (10 pts) $\displaystyle\sum_{m=1}^{\infty} \frac{m}{m^4+2}$

   **Solution:** Note that

$$\frac{m}{m^4+2} < \frac{m}{m^4}$$

   Since $\sum \frac{1}{m^3}$ is a convergent $p-$series, with $p = 3$, and

$$0 < \frac{m}{m^4+2} < \frac{1}{m^3}$$

   we may use direct comparison.

   By direct comparison to convergent $p-$series

$$\sum_{m=1}^{\infty} \frac{1}{m^3}$$

   this series **converges**.

   (b) (10 pts) $\displaystyle\sum_{k=2}^{\infty} \frac{[(k+1)!]^2}{(k^2-1)(k!)^2}$

   **Solution:** Consider the limit of the terms:

$$\lim_{k\to\infty} \frac{[(k+1)!]^2}{(k^2-1)(k!)^2} = \lim_{k\to\infty} \frac{(k+1)!\cdot(k+1)!}{(k-1)(k+1)(k!)(k!)} = \lim_{k\to\infty} \frac{(k+1)}{(k-1)}$$

   After simplification, we see that

$$\lim_{k\to\infty} \frac{k+1}{k-1} = 1$$

   Since $\lim_{k\to\infty} a_k \neq 0$, by the test for divergence, this **diverges.**

   (c) (10 pts) $\displaystyle\sum_{n=3}^{\infty} \frac{\ln(n^3)}{2n}$

   **Solution:** There are several approaches one may take to this problem. Note that

   $\ln(n^3) = 3\ln(n)$ by log laws. Then this may be written as $\displaystyle\sum_{n=3}^{\infty} \frac{3}{2}\frac{\ln(n)}{n}$.

Method 1: DCT. Note that for $n \geq 3, 0 < \frac{1}{n} < \frac{3\ln(n)}{2n}$, but $\sum \frac{1}{n}$ is the harmonic series, which diverges. Thus the original series diverges.

Method 2: LCT.

$$\lim_{n\to\infty} \frac{\frac{3\ln(n)}{2n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{3n\ln(n)}{2n} = \lim_{n\to\infty} \frac{3\ln(n)}{2} = \infty$$

Since the result is infinity, and we are limit comparing to the divergent series $\sum \frac{1}{n}$, the original series also diverges.

Method 3: Integral test. Note that $f(x) = \frac{3\ln(x)}{2x}$ is positive and continuous for $x \geq 3$, and

$$f'(x) = \frac{6 - 6\ln(x)}{4x^2}$$

is less than 0 for all $x > e$, so is definitely negative for $x \geq 3$. Therefore the function is decreasing. The function $f(x)$ can be used for the integral test.

$$\int_3^\infty \frac{3\ln(x)}{x} dx = \lim_{t\to\infty} \int_3^t \frac{3\ln(x)}{2x} dx$$

Let $u = \ln(x)$. Then $du = \frac{1}{x}dx$ nd the new bounds are $\ln(3)$ and $\ln(t)$.

$$\lim_{t\to\infty} \int_{\ln(3)}^{\ln(t)} \frac{3}{2} u du = \lim_{t\to\infty} \frac{3}{4} u^2 \Big|_{\ln(3)}^{\ln(t)} = \infty$$

Since the integral diverges, the series also **diverges.**

2. (34 pts) Consider the following series, which sums to $S$:

$$S = \sum_{n=1}^{\infty} \frac{(-3)^n}{4^n(n!)}$$

(a) (6 pts) Write out $s_1, s_2$, and $s_3$, the first three partial sums. You do not need to simplify your answer.

**Solution:**

$$s_1 = \frac{-3^1}{4^1(1!)} = -\frac{3}{4}$$

$$s_2 = -\frac{3}{4} + \frac{(-3)^2}{4^2(2!)} = -\frac{3}{4} + \frac{9}{32}$$

$$s_3 = -\frac{3}{4} + \frac{(-3)^2}{4^2(2!)} + \frac{(-3)^3}{4^3(3!)} = -\frac{3}{4} + \frac{9}{32} - \frac{27}{384}$$

Per the instructions, the unevaluated forms are acceptable, as are the versions where the exponentials and factorials are evaluated. Both are provided for convenience.

(b) (12 pts) State the hypotheses of the Alternating Series Estimation Theorem and show that each is satisfied.

**Solution:** Let $b_n = |a_n| = \left(\frac{3}{4}\right)^n \frac{1}{n!}$. The hypotheses are:

1) $b_n$ decreasing, i.e. $b_{n+1} < b_n$
2) $\lim_{n\to\infty} b_n = 0$

First we show decreasing. We can't take a derivative of a corresponding continuous function because of the factorial. We investigate by inequality instead.

$$\left(\frac{3}{4}\right)^{n+1} \frac{1}{(n+1)!} <^? \left(\frac{3}{4}\right)^n \frac{1}{n!}$$

$$\left(\frac{3}{4}\right) <^? (n+1) \qquad \text{Simplify}$$

$$-\frac{1}{4} <^? n$$

This is true for all $n \geq 0$. Thus the inequality on the first line is true for $n \geq 0$ and $b_n$ is decreasing.

Now we investigate the limit:

$$\lim_{n\to\infty} \left(\frac{3}{4}\right)^n \frac{1}{n!} = \lim_{n\to\infty} \left(\frac{3}{4}\right)^n \lim_{n\to\infty} \left(\frac{1}{n!}\right) \qquad \text{By limit laws}$$
$$= 0$$

As $n!$ is strictly increasing and going to infinity as $n \to \infty$, so $\frac{1}{n!}$ is strictly decreasing and going to 0 as $n \to \infty$. Also, since $\frac{3}{4} < 1$, we know that $\lim_{n\to\infty} \left(\frac{3}{4}\right)^n$ is 0 by our previous work with geometric sequences.

(There are multiple ways to justify this limit, but "$4^n(n!)$ grows much faster than $3^n$ " is an informal, and not formal, justification. To receive credit for a growth rate argument, the student should in some way argue *why* $4^n(n!)$ grows much faster than $3^n$, e.g. by expanding the factorial, etc., and then explain why that makes the limit 0.)

Thus the absolute value of the terms is decreasing with a limit of 0, and the hypotheses of the ASET are met.

(c) (6 pts) Find an estimate for the error if $s_3$ is used to approximate $S$. You do not need to simplify your answer.

**Solution:** We already showed ASET applies. We know that $|S - s_n| \leq b_{n+1}$, where $b_n$ is the same positive sequence used in the last part.

$$b_{3+1} = b_4 = \left(\frac{3}{4}\right)^4 \cdot \frac{1}{4!} = \frac{81}{256} \cdot \frac{1}{24}$$

Note: this is approximately 0.013 or $|S - s_3| < 2 * 10^{-2}$. Pretty good for only three terms!

(d) (10 pts) Does this series converge absolutely or conditionally?

**Solution:** We have already looked at the hypotheses of the Alternating Series Test in the previous part, so we know it converges at least conditionally. We must then investigate absolute convergence. We see that $|a_n| = \dfrac{3^n}{4^n(n!)}$.

Option 1: The factorial suggests ratio test would be a good choice. Then:

$$\begin{aligned} L &= \lim_{n\to\infty} \left| \frac{3^{n+1}}{4^{n+1}(n+1)!} \cdot \frac{4^n(n!)}{3^n} \right| \\ &= \lim_{n\to\infty} \left| \frac{3}{4(n+1)} \right| \\ L &= 0 \end{aligned}$$

Since $L = 0 < 1$, this series converges absolutely.

Option 2: We may use direct comparison to convergent geometric series $\sum (\frac{3}{4})^n$: since $0 < \dfrac{3^n}{4^n(n!)} \leq \left(\dfrac{3}{4}\right)^n$ for all $n \geq 0$, and $\displaystyle\sum_{n=1}^{\infty} \frac{3}{4} \cdot \left(\frac{3}{4}\right)^{n-1}$ is a convergent geometric series with $|r| < 1$, by direct comparison, the original series also converges.

3. (10 pts) Find the sum of the series, if it exists, for $\displaystyle\sum_{n=1}^{\infty} \frac{1+3^{n-1}}{5^n}$.

**Solution:** We may split this series into two:

$$\sum_{n=1}^{\infty} \frac{1+3^{n-1}}{5^n} = \sum_{n=1}^{\infty} \frac{1}{5^n} + \left(\frac{1}{5}\right) \cdot \frac{3^{n-1}}{5^{n-1}} = \sum_{n=1}^{\infty} \frac{1}{5} \cdot \left(\frac{1}{5}\right)^{n-1} + \sum_{n=1}^{\infty} \frac{1}{5} \cdot \left(\frac{3}{5}\right)^{n-1}$$

This is now the sum of two convergent geometric series, the first with $r = \frac{1}{5}$ and $a = \frac{1}{5}$, and the second with $r = \frac{3}{5}$ and $a = \frac{1}{5}$.

The sum of two convergent series converges. Using the sum of geometric series:

$$\frac{(1/5)}{1-(1/5)} + \frac{(1/5)}{1-(3/5)} = \frac{(1/5)}{(4/5)} + \frac{(1/5)}{(2/5)} = \frac{3}{4}$$

4. (10 pts) Consider the following power series:

$$\sum_{n=0}^{\infty} c_n (x+1)^n$$

This power series is convergent for $x = -2$, and divergent for $x = 5$.

Are the following statements about this power series true or false? Justify your answer.

(a) The series definitely diverges for $x = -7$.

(b) We need more information to say whether $\displaystyle\sum_{n=0}^{\infty} c_n$ diverges.

**Solution:**

(a) We see that this power series is centered at $x = -1$. We know it diverges for $x = 5$, but it is possible this is a divergent endpoint. Thus $R = 6$ is the largest possible radius of convergence. This would make $x = -7$ a possible endpoint. If it is an endpoint, the series may possibly converge at $x = -7$. Thus this is **false.**

(b) First, notice that $\displaystyle\sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty} c_n (0+1)^n$. We see that this is the given power series evaluated at $x = 0$.

Since the series converges for $x = -2$, we see that $R = 1$ is the smallest possible radius of convergence. This radius would make $x = 0$ an endpoint, which would mean it could diverge (or converge!).

However, $1 \leq R \leq 6$. For any radius of convergence in this range other than $R = 1$, $x = 0$ is within the interval of convergence. Thus is could *also* possibly converge.

Therefore this statement is **true.**

5. (16 pts) Find a power series representation for the following function:

$$f(x) = \frac{8}{8 + x^3}$$

Include the interval and radius of convergence for the power series.

**Solution:** With a little manipulation, we may see

$$\frac{8}{8+x^3} = \frac{8}{8(1+\frac{x^3}{8})} = \frac{1}{1-(-\frac{x^3}{8})} \text{ or } \frac{1}{1-(-(\frac{x}{2})^3)}$$

This may be represented as the sum of a power series with $r = (-\frac{1}{8}x^3)$ and $a = 1$:

$$\frac{1}{1-(-\frac{x^3}{8})} = \sum_{k=0}^{\infty}(-\frac{1}{8}x^3)^k = \sum_{k=0}^{\infty}\frac{(-1)^k}{8^k}x^{3k}$$

(Either form is acceptable).

The center, we can see explicitly in first form, is 0.

For the interval of convergence, we need $|r| < 1$. We solve the following inequality:

$$|-\frac{1}{8}x^3| < 1 \implies -1 < -\frac{1}{8}x^3 < 1$$
$$-8 < x^3 < 8$$
$$(-8)^{1/3} < x < 8^{1/3}$$
$$-2 < x < 2$$

Since this is geometric, the endpoints diverge. Thus:

$$\text{Interval} = (-2, 2),\ R = 2$$