1. (**22 pts**) Consider $g(x) = \int_{-1}^{-\sqrt{x+1}} \frac{1}{t^2+3t}\, \mathrm{d}t$

   (a) Find $g'(x)$

   (b) Find the equation of the tangent line to $g(x)$ when $x = 0$.

Solution:

(a) To find $g'(x)$, we use FTC part 1:

$$\begin{aligned} g'(x) &= \frac{d}{dx}\int_{-1}^{-\sqrt{x+1}} \frac{1}{t^2+3t}\, \mathrm{d}t \\ &= \frac{1}{(-\sqrt{x+1})^2 + 3(-\sqrt{x+1})} \cdot \frac{d}{dx}\left[-\sqrt{x+1}\right] \\ &= \frac{1}{x+1-3\sqrt{x+1}}\left(-\frac{1}{2}(x+1)^{-1/2}\right) \\ &= \boxed{\frac{-1}{(2\sqrt{x+1})(x+1-3\sqrt{x+1})}} \end{aligned}$$

(b) To find the equation of the tangent line, we need a point on the line and the slope. In part (a), we found $g'(x)$, so $g'(0)$ is our slope. In order to find the point, we find $g(0)$:

$$\begin{aligned} g(0) &= \int_{-1}^{-\sqrt{0+1}} \frac{1}{t^2+3t}\, \mathrm{d}t \\ &= \int_{-1}^{-1} \frac{1}{t^2+3t}\, \mathrm{d}t \\ &= 0 \end{aligned}$$

And thus, our point is $(0, 0)$. To find the slope, we evaluate $g'(0)$:

$$\begin{aligned} g'(0) &= \frac{-1}{(2\sqrt{0+1})(0+1-3\sqrt{0+1})} \\ &= \frac{1}{4} \end{aligned}$$

So, the equation of the tangent line to g at $x = 0$ is $\boxed{y = \frac{x}{4}}$

2. (**28 pts**) Consider $\displaystyle\int_0^{\pi} \cos^2(x)\,\mathrm{d}x$

(a) Compute R_3, the right endpoint approximation using 3 equispaced intervals.

(b) Write the integral as a limit of Riemann sums (do not evaluate the limit).

(c) Evaluate the integral. (Hint: you may want to use a trig identity).

Solution:

(a) To compute R_3, we find $\Delta x = \dfrac{b-a}{n} = \dfrac{\pi}{3}$. So, the right endpoints of our intervals are $x_1 = \dfrac{\pi}{3}$, $x_2 = \dfrac{2\pi}{3}$, and $x_3 = \pi$. Thus, if $f(x) = \cos^2(x)$,

$$\begin{aligned} R_3 &= \Delta x[f(x_1) + f(x_2) + f(x_3)] \\ &= \frac{\pi}{3}\left[\left(\frac{1}{2}\right)^2 + \left(\frac{-1}{2}\right)^2 + (-1)^2\right] \\ &= \frac{\pi}{3}\cdot\frac{3}{2} \\ &= \boxed{\frac{\pi}{2}} \end{aligned}$$

(b) To write the integral as a limit of Riemann sums, we recall that a Riemann sum is of the form: $\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$.

We note that $\Delta x = \dfrac{b-a}{n} = \dfrac{\pi}{n}$. Since $a = 0$ and $b = \pi$, $x_i = a + i\Delta x = \dfrac{i\pi}{n}$. Thus, the integral can be written:

$$\boxed{\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\pi}{n}\cos^2\left(\frac{i\pi}{n}\right)}$$

(c) To compute the integral, we use the double angle formula to re-write $\cos^2(x) = \dfrac{\cos(2x)+1}{2}$. Thus:

$$\begin{aligned} \int_0^{\pi}\cos^2(x)\,\mathrm{d}x &= \int_0^{\pi}\frac{\cos(2x)+1}{2}\,\mathrm{d}x \\ &= \int_0^{\pi}\frac{\cos(2x)}{2}\,\mathrm{d}x + \int_0^{\pi}\frac{1}{2}\,\mathrm{d}x \end{aligned}$$

To compute the first integral, we use the u-substitution $u = 2x$:

$$\begin{aligned}
\int_0^{\pi} \frac{\cos(2x)}{2}\,\mathrm{d}x &= \frac{1}{2}\int_{u=0}^{u=2\pi} \cos(u)\frac{\mathrm{d}u}{2} \\
&= \left[-\frac{1}{4}\sin(u)\right]_0^{2\pi} \\
&= -\frac{1}{4}(0-0) \\
&= 0
\end{aligned}$$

To evaluate the second integral:

$$\begin{aligned}
\int_0^{\pi} \frac{1}{2}\,\mathrm{d}x &= \left[\frac{1}{2}x\right]_0^{\pi} \\
&= \frac{1}{2}(\pi - 0) \\
&= \frac{\pi}{2}.
\end{aligned}$$

So, the value of $\displaystyle\int_0^{\pi} \cos^2(x)\,\mathrm{d}x = \boxed{\frac{\pi}{2}}$, which happens to be equal to R_3.

3. (**15 pts**) The acceleration of gravity on Mars is about $\frac{15}{4}$ m/s^2. The height of Olympus Rupes, the cliff surrounding Olympus Mons (the largest volcano in our solar system) can reach 7,500 m. Suppose an astronaut drops a rock off the highest point on the Olympus Rupes cliff. Answer the following questions.

   (a) How long will it take for the rock to hit the ground?

   (b) What is the velocity of the rock when it hits the ground?

Solution: We start by writing down what is given in the problem statement. Note that we are told $a(t) = \frac{-15}{4}$ (taking down to be the negative direction). We also know the rock is dropped, so $v(0) = 0$. The height of the cliff is 7500, so $s(0) = 7,500$. Let t_f be the time (in seconds) that the rock hits the ground. Therefore, $s(t_f) = 0$. We will use this information to solve the problems.

(a) In order to determine how long it will take for the rock to hit the ground, we find the velocity function, which is the antiderivative of $a(t)$. So, $v(t) = \frac{-15}{4}t + C$. Since we are given that $v(0) = 0$, we know $C = 0$.

Therefore, we know $v(t) = -\frac{15}{4}t$, and we can take the antiderivative of v to find the position function $s(t) = -\frac{15}{8}t^2 + D$. Again, we know $s(0) = 7500$, which means $D = 7500$. We are now able to solve $s(t_f) = -\frac{15}{8}(t_f)^2 + 7500 = 0$ to find how long it takes for the rock to hit the ground:

$$s(t_f) = 0 = -\frac{15}{8}(t_f)^2 + 7500$$

$$7500 = \frac{15}{8}(t_f)^2$$

$$4000 = (t_f)^2$$

$$\pm 20\sqrt{10} = t_f$$

$$t_f = \boxed{20\sqrt{10} \text{ sec}}$$

(b) To find the velocity of the rock when it hits the ground, we plug in $t_f = 20\sqrt{10}$ to $v(t)$:

$$v(t_f) = -\frac{15}{4}(20\sqrt{10})$$

$$= -15 \cdot 5\sqrt{10}$$

$$= \boxed{-75\sqrt{10} \text{ m/s}}$$

4. (**15 pts**) Consider $f(x) = x^3 + 5x^2 - 1$.

(a) Show that Newton's method will fail to converge to a root of this function with an initial guess of $x_0 = \dfrac{-10}{3}$.

(b) Find x_1 if $x_0 = -1$ (first iteration of Newton's method given an initial guess).

Solution:

(a) An iteration of Newton's method is: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$. Assuming that $x_0 = -10/3$, we will attempt to find x_1.

Given $f(x) = x^3 + 5x^2 - 1$, we compute $f'(x) = 3x^2 + 10x$. Now we see that $f'(x_0) = 3\left(\frac{100}{9}\right) - 10\left(\frac{10}{3}\right) = 0$. This means there is a horizontal tangent at $x = -10/3$, so Newton's method will not work with this initial guess.

(b) To find x_1, we find

$$f(x_0) = (-1)^3 + 5(-1)^2 - 1 = -1 + 5 - 1 = 3$$

$$f'(x_0) = 3(-1)^2 + 10(-1) = 3 - 10 = -7$$

Now, we plug those values into Newton's method formula:

$x_1 = -1 - (3/(-7)) = -1 + 3/7 = \boxed{-4/7}$

5. (**20 pts**) Evaluate the following:

(a) $\displaystyle\int_0^2 |x-2| - \sqrt{4-x^2}\,\mathrm{d}x$. Hint: use geometry.

(b) $\displaystyle\int \sqrt{x}\sin\left(1+\sqrt{x^3}\right)\mathrm{d}x$

Solution:

(a) First we note that this integral can be broken up into 2 separate integrals:

$$\int_0^2 |x-2| - \sqrt{4-x^2}\,\mathrm{d}x = \int_0^2 |x-2|\,\mathrm{d}x - \int_0^2 \sqrt{4-x^2}\,\mathrm{d}x$$

The first integral gives us the area of a triangle with height and base both 2, above the x axis, so the area between the curve and the x axis is 2.

The second integral is the area of a quarter circle of radius 2 centered at the origin, which is $\frac{1}{4}4\pi = \pi$. So, the whole integral is $\boxed{2-\pi}$

(b) To compute this integral, we use the u-substitution $u = 1 + x^{3/2}$. Thus, $\mathrm{d}u = \frac{3}{2}x^{1/2}\,\mathrm{d}x \implies \mathrm{d}x = \frac{2\mathrm{d}u}{3x^{1/2}}$. Plugging in:

$$\begin{aligned}
\int \sqrt{x}\sin\left(1+\sqrt{x^3}\right)\mathrm{d}x &= \int \sqrt{x}\sin(u)\,\frac{2\,\mathrm{d}u}{3x^{1/2}} \\
&= \frac{2}{3}\int \sin(u)\,\mathrm{d}u \\
&= \frac{2}{3}[-\cos(u)] + C \\
&= \boxed{-\frac{2}{3}\cos\left(1+\sqrt{x^3}\right) + C}
\end{aligned}$$