1. [2360/071125 (20 pts)] Consider the differential equation $y'' + \frac{y'}{t} = \frac{1}{t^2 \ln t}$, $t > 0$.

   (a) (8 pts) Prove that the set $\{1, \ln t\}$ forms a basis for the solution space of the associated homogeneous equation on the given interval. Fully justify your answer.

   (b) (12 pts) Find the general solution of the differential equation. u-substitution may prove beneficial.

SOLUTION:

(a) Begin by showing that both functions are solutions to the differential equation:

$$(1)'' + \frac{(1)'}{t} = 0 + 0 = 0$$
$$(\ln t)'' + \frac{(\ln t)'}{t} = \left(-\frac{1}{t^2}\right) + \frac{1}{t^2} = 0$$

Check for linear independence.

$$W[1, \ln t](t) = \begin{vmatrix} 1 & \ln t \\ 0 & \frac{1}{t} \end{vmatrix} = \frac{1}{t} \neq 0 \quad \text{for } t > 0$$

implying that the functions do form a basis for the solution space of the associated homogeneous equation since there are two linearly independent solutions and the differential equation is order two.

(b) We must use variation of parameters with $y_p = v_1 y_1 + v_2 y_2$ to solve the nonhomogeneous equation. With $y_1 = 1, y_2 = \ln t$ and $f(t) = \frac{1}{t^2 \ln t}$ we have

$$v_1 = \int \frac{-y_2 f}{W[1, \ln t]}\,\mathrm{d}t = -\int \frac{\ln t(1/t^2 \ln t)}{1/t}\,\mathrm{d}t = -\int \frac{1}{t}\,\mathrm{d}t = -\ln|t| = -\ln t \qquad \text{since } t > 0$$
$$v_2 = \int \frac{y_1 f}{W[1, \ln t]}\,\mathrm{d}t = \int \frac{1(1/t^2 \ln t)}{1/t}\,\mathrm{d}t = \int \frac{1}{t \ln t}\,\mathrm{d}t \stackrel{u=\ln t}{=} \ln|\ln t|$$

Thus

$$y = y_h + y_p = c_1 + \tilde{c}_2 \ln t - \ln t + \ln t\,(\ln|\ln t|) = c_1 + c_2 \ln t + \ln t\,(\ln|\ln t|)$$

where the last equality follows from the fact that we can combine the $\tilde{c}_2$ and -1 into a single new constant multiplying $\ln t$. ■

2. [2360/071125 (43 pts)] An harmonic oscillator is governed by the initial value problem

$$2\ddot{x} + 6\dot{x} + 4x = 80\cos 2t,\ x(0) = 1,\ \dot{x}(0) = 1 \tag{1}$$

   (a) (20 pts) Find the position of the mass when $t = \pi$?

   (b) (4 pts) What is the velocity of the mass when $t = \pi$?

   (c) (4 pts) Identify the transient and steady state solutions, if they exist.

   (d) (5 pts) If the oscillator is unforced, but nothing else changes, find the time(s), if any, when the graph of the solution crosses the t-axis. Assume the time takes on only nonnegative values.

   (e) (5 pts) If the mass and the forcing function remain the same, write the differential equation that would describe the oscillator in resonance. You need not supply any initial conditions nor solve the resulting differential equation.

   (f) (5 pts) How would you modify Equation (1) so that it would model a conservative system, that is, one in which the energy is constant?

SOLUTION:

(a) The characteristic equation of the associated homogeneous equation is $2r^2 + 6r + 4 = 2(r^2 + 3r + 2) = 2(r+1)(r+2) = 0$ with roots $r = -1, -2$ so that $x_h = c_1 e^{-t} + c_2 e^{-2t}$.

The particular solution has the form $x_p = A\cos 2t + B\sin 2t$. Substituting this into the differential equation gives

$$(-4A + 12B)\cos 2t + (-12A - 4B)\sin 2t = 80\cos 2t$$

Equating coefficients on either side of this equation yields the linear system

$$\begin{aligned} -4A + 12B &= 80 \\ -12A - 4B &= 0 \end{aligned}$$

whose solution is $A = -2, B = 6$. Thus $x_p = -2\cos 2t + 6\sin 2t$ and $x = x_h + x_p = c_1 e^{-t} + c_2 e^{-2t} - 2\cos 2t + 6\sin 2t$.

Now apply the initial conditions.

$$\begin{aligned} x(0) &= c_1 + c_2 - 2 = 1 \implies c_1 + c_2 = 3 \\ \dot{x}(0) &= -c_1 - 2c_2 + 12 = 1 \implies -c_1 - 2c_2 = -11 \end{aligned}$$

giving $c_1 = -5$ and $c_2 = 8$ so that the displacement is $x(t) = 8e^{-2t} - 5e^{-t} - 2\cos 2t + 6\sin 2t$. The position of the mass at $t = \pi$ is

$$x(\pi) = 8e^{-2\pi} - 5e^{-\pi} - 2$$

(b) The velocity at time t is $\dot{x}(t) = -16e^{-2t} + 5e^{-t} + 4\sin 2t + 12\cos 2t$ so at $t = \pi$ we have

$$\dot{x}(\pi) = 5e^{-\pi} - 16e^{-2\pi} + 12$$

(c)

$$\begin{aligned} x_{\text{transient}} &= 8e^{-2t} - 5e^{-t} \\ x_{\text{steady state}} &= 6\sin 2t - 2\cos 2t \end{aligned}$$

(d) The general solution in this case is $x_h = c_1 e^{-t} + c_2 e^{-2t}$. Applying the initial conditions yields $c_1 = 3$ and $c_2 = -2$. Thus $x(t) = 3e^{-t} - 2e^{-2t}$. The solution will cross the t-axis if/when this vanishes.

$$\begin{aligned} 3e^{-t} - 2e^{-2t} &= 0 \\ e^{-2t}\left(3e^{t} - 2\right) &= 0 \\ e^{t} &= \frac{2}{3} \\ t &= \ln\frac{2}{3} \end{aligned}$$

Since we assume only values of $t \geq 0$, the solution does not cross the t-axis and the mass in the oscillator does not pass through its rest position.

(e) To be in resonance, the oscillator must be undamped ($b = 0$) and the circular frequency ($\omega_0 = \sqrt{k/2}$) must be the same as the frequency of the forcing function ($\omega_f = 2$) implying that $k = 8$. The differential equation is then $2\ddot{x} + 8x = 80\cos 2t$.

(f) $2\ddot{x} + 4x = 0$. No change needed to the initial conditions.

■

3. [2360/071125 (9 pts)] Convert the initial value problem $y^{(4)} + 2y' - 5y = 0$, $y(1) = 1$, $y'(1) = -2$, $y''(1) = 3$, $y'''(1) = -4$ into a system of first order differential equations with appropriate initial condition(s). Write your answer in terms of matrices, if possible. If not possible, explain why not.

SOLUTION:

Let $u_1 = y, u_2 = y', u_3 = y'', u_4 = y'''$. Then

$$\begin{aligned} u_1' &= y' = u_2 \\ u_2' &= y'' = u_3 \\ u_3' &= y''' = u_4 \\ u_4' &= y^{(4)} = -2y' + 5y = -2u_2 + 5u_1 \end{aligned}$$

with $u_1(1) = 1, u_2(1) = -2, u_3(1) = 3, u_4(1) = -4$. Written as matrices we have

$$\begin{bmatrix} u_1' \\ u_2' \\ u_3' \\ u_4' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 5 & -2 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}, \quad \begin{bmatrix} u_1(1) \\ u_2(1) \\ u_3(1) \\ u_4(1) \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 3 \\ -4 \end{bmatrix}$$

■

4. [2360/071125 (10 pts)] Consider the differential equation $y'' - y' - 6y = f(t)$. For each of the following functions, write the form of the particular solution that would be used for solving the nonhomogeneous equation using the method of undetermined coefficients. **DO NOT** solve for the coefficients.

   (a) $f(t) = t^2 - 1$

   (b) $f(t) = 7e^{-2t}$

   (c) $f(t) = t^3 e^{3t}$

   (d) $f(t) = e^{-4t} - 9$

   (e) $f(t) = 4\cos 2t + 2e^{-2t}\sin 4t$

   SOLUTION:

   The characteristic equation of the associated homogeneous equation is $r^2 - r - 6 = (r-3)(r+2) = 0$ with roots $r = -2, 3$ giving a basis for the solution space of $\left\{e^{-2t}, e^{3t}\right\}$

   (a) $y_p = At^2 + Bt + C$

   (b) $y_p = Ate^{-2t}$

   (c) $y_p = t(At^3 + Bt^2 + Ct + D)e^{3t} = (At^4 + Bt^3 + Ct^2 + Dt)e^{3t}$

   (d) $y_p = Ae^{-4t} + B$

   (e) $y_p = A\cos 2t + B\sin 2t + Ce^{-2t}\cos 4t + De^{-2t}\sin 4t$

■

5. [2360/071125 (10 pts)] The following problems are not related.

   (a) (5 pts) Find a basis for the solution space of the differential equation

   $$y^{(5)} - 4y^{(4)} + 13y''' = 0$$

   (b) (5 pts) Find a fourth order, constant coefficient, homogeneous linear differential equation whose general solution is

   $$y(t) = c_1 e^{2t} + c_2 te^{2t} + c_3 \cos t + c_4 \sin t$$

   SOLUTION:

   (a) The characteristic equation is $r^5 - 4r^4 + 13r^3 = r^3(r^2 - 4r + 13) = 0$ with the roots $2 \pm 3i$ each with multiplicity 1, and 0 with multiplicity 3. A basis is $\left\{1, t, t^2, e^{2t}\cos 3t, e^{2t}\sin 3t\right\}$

   (b) From the general solution, we can conclude that the roots of the characteristic equation are 2 with multiplicity 2 and $\pm i$. This gives a characteristic equation of $(r-2)^2(r^2+1) = r^4 - 4r^3 + 5r^2 - 4r + 4 = 0$. From this then we have the differential equation

   $$y^{(4)} - 4y''' + 5y'' - 4y' + 4y = 0$$

■

6. [2360/071125 (8 pts)] Find $\mathscr{L}^{-1}\left\{\dfrac{8s}{4s^2+1}\right\}$.

   SOLUTION:

$$\mathscr{L}^{-1}\left\{\frac{8s}{4s^2+1}\right\}=\mathscr{L}^{-1}\left\{\frac{8s}{4s^2+1}\left(\frac{\frac{1}{4}}{\frac{1}{4}}\right)\right\}=\mathscr{L}^{-1}\left\{\frac{2s}{s^2+\frac{1}{4}}\right\}=2\mathscr{L}^{-1}\left\{\frac{s}{s^2+\left(\frac{1}{2}\right)^2}\right\}=2\cos\frac{t}{2}$$

■