- This exam is worth 100 points and has 6 problems.
- Show all work and simplify your answers! Answers with no justification will receive no points unless otherwise noted.
- Begin each problem on a new page.
- **DO NOT LEAVE THE EXAM UNTIL YOUR HAVE SATISFACTORILY SCANNED AND UPLOADED YOUR EXAM TO GRADESCOPE.**
- You are taking this exam in a proctored and honor code enforced environment. No calculators, cell phones, or other electronic devices or the internet are permitted during the exam. You are allowed one 8.5"× 11" crib sheet with writing on one side.
- Remote students are allowed use of a computer during the exam only for a live video of their hands and face and to view the exam in the Zoom meeting. Remote students cannot interact with anyone except the proctor during the exam.

0. At the top of the first page that you will be scanning and uploading to Gradescope, write the following statement and sign your name to it: "I will abide by the CU Boulder Honor Code on this exam." **FAILURE TO INCLUDE THIS STATEMENT AND YOUR SIGNATURE MAY RESULT IN A PENALTY.**

1. [2360/071125 (20 pts)] Consider the differential equation $y'' + \dfrac{y'}{t} = \dfrac{1}{t^2 \ln t}$, $t > 0$.

   (a) (8 pts) Prove that the set $\{1, \ln t\}$ forms a basis for the solution space of the associated homogeneous equation on the given interval. Fully justify your answer.

   (b) (12 pts) Find the general solution of the differential equation. $u$-substitution may prove beneficial.

2. [2360/071125 (43 pts)] An harmonic oscillator is governed by the initial value problem

$$2\ddot{x} + 6\dot{x} + 4x = 80\cos 2t,\ x(0) = 1,\ \dot{x}(0) = 1 \tag{1}$$

   (a) (20 pts) Find the position of the mass when $t = \pi$?

   (b) (4 pts) What is the velocity of the mass when $t = \pi$?

   (c) (4 pts) Identify the transient and steady state solutions, if they exist.

   (d) (5 pts) If the oscillator is unforced, but nothing else changes, find the time(s), if any, when the graph of the solution crosses the $t$-axis. Assume the time takes on only nonnegative values.

   (e) (5 pts) If the mass and the forcing function remain the same, write the differential equation that would describe the oscillator in resonance. You need not supply any initial conditions nor solve the resulting differential equation.

   (f) (5 pts) How would you modify Equation (1) so that it would model a conservative system, that is, one in which the energy is constant?

3. [2360/071125 (9 pts)] Convert the initial value problem $y^{(4)} + 2y' - 5y = 0$, $y(1) = 1$, $y'(1) = -2$, $y''(1) = 3$, $y'''(1) = -4$ into a system of first order differential equations with appropriate initial condition(s). Write your answer in terms of matrices, if possible. If not possible, explain why not.

4. [2360/071125 (10 pts)] Consider the differential equation $y'' - y' - 6y = f(t)$. For each of the following functions, write the form of the particular solution that would be used for solving the nonhomogeneous equation using the method of undetermined coefficients. **DO NOT** solve for the coefficients.

   (a) $f(t) = t^2 - 1$

   (b) $f(t) = 7e^{-2t}$

   (c) $f(t) = t^3 e^{3t}$

   (d) $f(t) = e^{-4t} - 9$

   (e) $f(t) = 4\cos 2t + 2e^{-2t}\sin 4t$

**MORE PROBLEMS AND LAPLACE TRANSFORM TABLE BELOW/ON REVERSE**

5. [2360/071125 (10 pts)] The following problems are not related.

(a) (5 pts) Find a basis for the solution space of the differential equation

$$y^{(5)} - 4y^{(4)} + 13y''' = 0$$

(b) (5 pts) Find a fourth order, constant coefficient, homogeneous linear differential equation whose general solution is

$$y(t) = c_1 e^{2t} + c_2 t e^{2t} + c_3 \cos t + c_4 \sin t$$

6. [2360/071125 (8 pts)] Find $\mathscr{L}^{-1}\left\{\dfrac{8s}{4s^2+1}\right\}$.

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**Short table of Laplace Transforms:** $\mathscr{L}\{f(t)\} = F(s) \equiv \displaystyle\int_0^\infty e^{-st} f(t)\,\mathrm{d}t$

In this table, $a, b, c$ are real numbers with $c \geq 0$, and $n = 0, 1, 2, 3, \ldots$

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$$\mathscr{L}\left\{t^n e^{at}\right\} = \frac{n!}{(s-a)^{n+1}} \qquad \mathscr{L}\left\{e^{at}\cos bt\right\} = \frac{s-a}{(s-a)^2+b^2} \qquad \mathscr{L}\left\{e^{at}\sin bt\right\} = \frac{b}{(s-a)^2+b^2}$$

$$\mathscr{L}\{\cosh bt\} = \frac{s}{s^2-b^2} \qquad \mathscr{L}\{\sinh bt\} = \frac{b}{s^2-b^2}$$

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{\mathrm{d}^n F(s)}{\mathrm{d}s^n} \qquad \mathscr{L}\left\{e^{at} f(t)\right\} = F(s-a) \qquad \mathscr{L}\{\delta(t-c)\} = e^{-cs}$$

$$\mathscr{L}\{t f'(t)\} = -F(s) - s\frac{\mathrm{d}F(s)}{\mathrm{d}s} \qquad \mathscr{L}\{f(t-c)\,\text{step}(t-c)\} = e^{-cs}F(s) \qquad \mathscr{L}\{f(t)\,\text{step}(t-c)\} = e^{-cs}\mathscr{L}\{f(t+c)\}$$

$$\mathscr{L}\left\{f^{(n)}(t)\right\} = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \cdots - f^{(n-1)}(0)$$

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