1. (**20 pts**) Consider the equation $y^2 - xy = 3\sqrt{x} - 1$.

   (a) Find $\dfrac{dy}{dx}$.

   (b) Find the equation of the tangent line to the curve when $x = 4$ and $y > 0$.

Solution:

(a) We differentiate both sides of the equation implicitly with respect to x:

$$\frac{d}{dx}(y^2 - xy) = \frac{d}{dx}(3\sqrt{x} - 1)$$

On the LHS, we need to use the chain and product rules and on the RHS we need to use the power rule.

$$\begin{aligned} 2y\frac{dy}{dx} - \left(x\frac{dy}{dx} + y\right) &= \frac{3}{2\sqrt{x}} - 0 \\ 2y\frac{dy}{dx} - x\frac{dy}{dx} - y &= \frac{3}{2\sqrt{x}} \\ (2y - x)\frac{dy}{dx} &= \frac{3}{2\sqrt{x}} + \frac{2y\sqrt{x}}{2\sqrt{x}} \\ \frac{dy}{dx} &= \boxed{\frac{3 + 2y\sqrt{x}}{2\sqrt{x}(2y - x)}} \end{aligned}$$

(b) To find the equation of the tangent line, we need a point on the line and the slope. We must find the y coordinate of the point using the original equation with $x = 4$ plugged in:

$$\begin{aligned} y^2 - 4y &= 3\sqrt{4} - 1 \\ y^2 - 4y &= 5 \\ y^2 - 4y - 5 &= 0 \\ (y - 5)(y + 1) &= 0 \implies y = 5 \text{ or } y = -1 \end{aligned}$$

Since $y > 0$, we take $y = 5$.

Now we plug $x = 4$, $y = 5$ into the derivative we found in part (a) to get the slope:

$$\begin{aligned} \left.\frac{dy}{dx}\right|_{(4,5)} &= \frac{3 + 2 \cdot 5\sqrt{4}}{2\sqrt{4}(2 \cdot 5 - 4)} \\ &= \frac{3 + 20}{4(6)} = \frac{23}{24} \end{aligned}$$

And we can now write our tangent line equation in point-slope form:

$$\boxed{y - 5 = \frac{23}{24}(x - 4)}$$

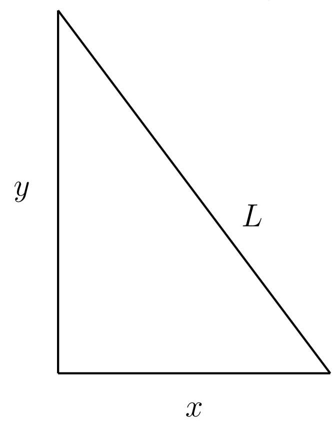
2. (**15 pts**) The top of a ladder slides down a vertical wall at a rate of 1/3 ft/s. At the moment when the bottom of the ladder is 5ft from the wall, it is sliding along the ground at a rate of 4/5 ft/s.

   (a) How high is the top of the ladder from the ground at the instant that the base is 5ft from the wall?

   (b) How long is the ladder?

Solution:

(a) This is a related rates problem where we are given all of the rates and asked to find the distance y from the ground to the top of the ladder at a given instant and, subsequently, the length of the ladder. A diagram for this problem is shown below. We note that x and y are both changing with time while L (the length of the ladder) is fixed.



Given:

$$\frac{dy}{dt} = -\frac{1}{3} \text{ ft/s}$$
$$\frac{dx}{dt} = \frac{4}{5} \text{ ft/s when } x = 5$$

Find: y when $x = 5$

Equation: $x^2 + y^2 = L^2$

Differentiate (d/dt):

$$\frac{d}{dt}(x^2 + y^2) = \frac{d}{dt}L^2$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \implies x\frac{dx}{dt} + y\frac{dy}{dt} = 0$$

Plug in values:

$$5\left(\frac{4}{5}\right) + y\left(-\frac{1}{3}\right) = 0$$

Solve for y:

$$4 = \frac{y}{3} \implies \boxed{y = 12 \text{ ft}}$$

(b) Now we can just plug in x and y to solve for L:

$$x^2 + y^2 = L^2$$
$$5^2 + 12^2 = L^2 \implies \boxed{L = 13 \text{ ft}}$$

3. (**40 pts**) Consider $f(x) = \cos x + \dfrac{x}{2}$ on the interval $[0, 2\pi]$.

(a) Find the intervals over which f is increasing and decreasing.

(b) Find and classify all local extrema (coordinate pairs) if any.

(c) Find the absolute maximum and minimum values attained by the function on $[0, 2\pi]$.

(d) Find the intervals of concavity of f.

(e) Find all points of inflection (coordinate pairs) if any.

(f) Sketch a graph of $y = f(x)$. Be sure to label any key points including extrema and points of inflection.

Solution:

(a) We must calculate the first derivative and set it equal to 0 to find the critical points of f:

$$f'(x) = -\sin x + \frac{1}{2}$$

$$0 = -\sin x + \frac{1}{2}$$

$$\sin x = \frac{1}{2} \implies x = \frac{\pi}{6}, \frac{5\pi}{6} \text{ for } x \text{ in } [0, 2\pi]$$

Use a sign chart or test values in each interval:

- On $[0, \frac{\pi}{6})$: $\sin x < \frac{1}{2} \Rightarrow f'(x) > 0 \rightarrow$ Increasing
- On $(\frac{\pi}{6}, \frac{5\pi}{6})$: $\sin x > \frac{1}{2} \Rightarrow f'(x) < 0 \rightarrow$ Decreasing
- On $(\frac{5\pi}{6}, 2\pi]$: $\sin x < \frac{1}{2} \Rightarrow f'(x) > 0 \rightarrow$ Increasing

Therefore, f is $\boxed{\text{increasing on } \left[0, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, 2\pi\right]}$ and $\boxed{\text{decreasing on } \left(\frac{\pi}{6}, \frac{5\pi}{6}\right)}$

(b) From part (a) we know there is a local max at $x = \dfrac{\pi}{6}$ and a local min at $x = \dfrac{5\pi}{6}$, so we need to plug in these x values to get the y-coordinates of the local extrema:

$$\begin{aligned} f\left(\frac{\pi}{6}\right) &= \cos\left(\frac{\pi}{6}\right) + \frac{\pi}{12} \\ &= \frac{\sqrt{3}}{2} + \frac{\pi}{12} = \frac{6\sqrt{3} + \pi}{12} \\ f\left(\frac{5\pi}{6}\right) &= \cos\left(\frac{5\pi}{6}\right) + \frac{5\pi}{12} \\ &= -\frac{\sqrt{3}}{2} + \frac{5\pi}{12} = \frac{5\pi - 6\sqrt{3}}{12} \end{aligned}$$

So, f has a $\boxed{\text{local max at } \left(\frac{\pi}{6}, \frac{6\sqrt{3} + \pi}{12}\right)}$ and a $\boxed{\text{local min at } \left(\frac{5\pi}{6}, \frac{5\pi - 6\sqrt{3}}{12}\right)}$

(c) To find absolute extrema, we just need to check the endpoints because we already evaluated the function at the critical numbers in part (b).

$$f(0) = \cos(0) + 0 = 1$$
$$f(2\pi) = \cos(2\pi) + \pi = 1 + \pi$$

We also note that $\pi \approx 3.14$, $\sqrt{3} \approx 1.7$, which we will use to compute the relative sizes of our local extrema. So,

$$f\left(\frac{\pi}{6}\right) = \frac{6\sqrt{3}+\pi}{12} \approx \frac{6(1.7)+3.14}{12} = \frac{13.34}{12}$$
$$f\left(\frac{5\pi}{6}\right) = \frac{5\pi-6\sqrt{3}}{12} \approx \frac{5(3.14)-6(1.7)}{12} = \frac{5.5}{12}$$

Thus, $f\left(\frac{\pi}{6}\right)$ is a bit larger than 1 and $f\left(\frac{5\pi}{6}\right)$ is a bit smaller than $\frac{1}{2}$.

Therefore, the $\boxed{\text{absolute max occurs at } (2\pi, \pi+1)}$

and the $\boxed{\text{absolute min is at } \left(\frac{5\pi}{6}, \frac{5\pi-6\sqrt{3}}{12}\right)}$

(d) We must calculate the second derivative and set it equal to 0 to find candidates for inflection points and intervals of concavity:

$$f''(x) = -\cos x$$
$$0 = -\cos x$$
$$\cos x = 0 \implies x = \frac{\pi}{2}, \frac{3\pi}{2} \text{ for } x \text{ in } [0, 2\pi]$$

Use a sign chart for $f''(x)$:

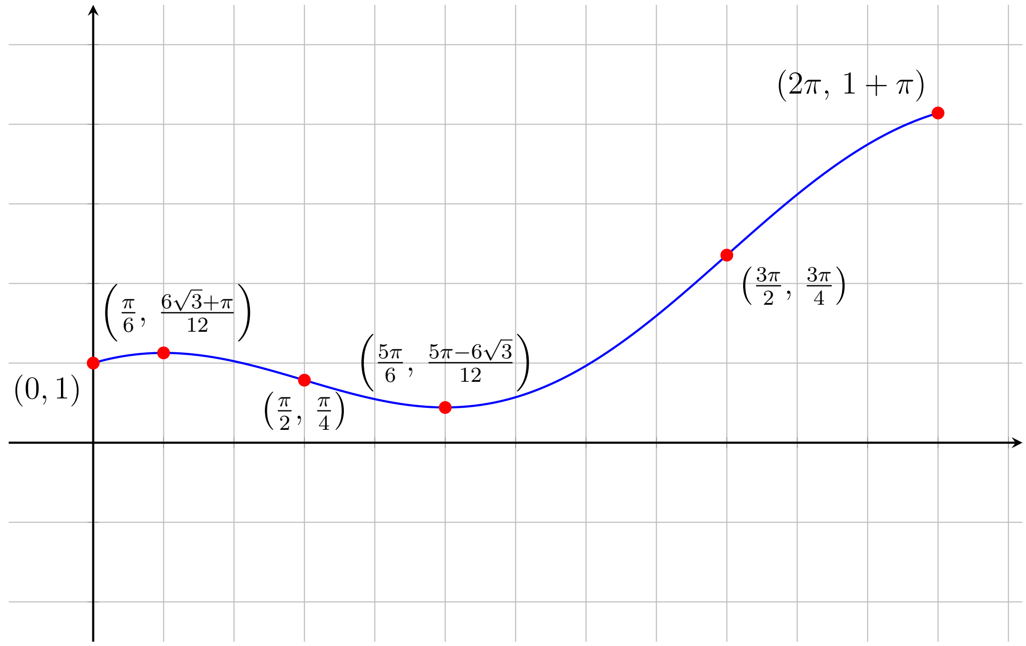
- $(0, \frac{\pi}{2})$: $\cos x > 0 \Rightarrow f''(x) < 0 \rightarrow$ concave down
- $(\frac{\pi}{2}, \frac{3\pi}{2})$: $\cos x < 0 \Rightarrow f''(x) > 0 \rightarrow$ concave up
- $(\frac{3\pi}{2}, 2\pi)$: $\cos x > 0 \Rightarrow f''(x) < 0 \rightarrow$ concave down

(e) From part (d), we know there are inflection points where concavity changes and $f''(x) = 0$, which happens at $x = \dfrac{\pi}{2}$ and $x = \dfrac{3\pi}{2}$

$$f\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) + \frac{\pi}{4} = \frac{\pi}{4}$$
$$f\left(\frac{3\pi}{2}\right) = \cos\left(\frac{3\pi}{2}\right) + \frac{3\pi}{4} = \frac{3\pi}{4}$$

Therefore, inflection points occur at $\boxed{\left(\frac{\pi}{2}, \frac{\pi}{4}\right) \text{ and } \left(\frac{3\pi}{2}, \frac{3\pi}{4}\right)}$

(f) And, finally, the graph of f on $[0, 2\pi]$:



4. (**25 pts**) Consider $g(x) = \dfrac{1}{(x+2)^k}$ for some $k > 0$.

(a) i. Find the linearization of g centered at $a = 1$.

ii. Use your linearization to estimate the value of $\dfrac{1}{3.01^3}$.

iii. Do you expect your estimate to be greater or less than the true value of $\dfrac{1}{3.01^3}$? Justify your answer.

(b) Let $k = 4$. Show that there is no value of c that satisfies the conclusion of the Mean Value Theorem on $[-4, 0]$. Explain, using the hypotheses, why this is not a contradiction of MVT.

Solution:

(a) i. To find the linearization of $g(x) = \dfrac{1}{(x+2)^k}$ centered at $a = 1$, we recall that

$$g(x) \approx L(x) = g(a) + g'(a)(x-a)$$

Note that $g(x) = (x+2)^{-k}$ and therefore $g'(x) = -k(x+2)^{-k-1}$. Plugging in $a = 1$:

$$g(a) = g(1) = 3^{-k} = \frac{1}{3^k}$$
$$g'(a) = g'(1) = -k(3)^{-k-1} = \frac{-k}{3^{k+1}}$$

Now we can set up $L(x)$ for the general k case:

$$\boxed{L(x) = \frac{1}{3^k} - \frac{k}{3^{k+1}}(x-1)}$$

ii. To estimate $\dfrac{1}{3.01^3}$, we note that $k = 3$ and $x = 1.01$:

$$\begin{aligned} L(x) &= \frac{1}{3^3} - \frac{3}{3^4}(x-1) \\ L(1.01) &= \frac{1}{27} - \frac{1}{27}\left(\frac{1}{100}\right) \\ g(1.01) = \frac{1}{3.01^3} \approx L(1.01) &= \frac{99}{2700} = \boxed{\frac{11}{300}} \end{aligned}$$

iii. To determine if our estimate is an over or under estimate, we can either sketch the graph or take the second derivative to determine that for $x > -2$ the graph is concave up, and thus the tangent lines are below the curve. Therefore, we expect that the linearization estimate is $\boxed{\text{an underestimate}}$ for the true value of $\dfrac{1}{3.01^3}$.

(b) To illustrate that there is no value of c that satisfies MVT on $[-4, 0]$ for $g(x) = \dfrac{1}{(x+2)^4}$, we must set up the conclusion of MVT assuming that there is such a value of c:

$$g'(c) = \frac{g(b) - g(a)}{b - a}$$

Note that for $k = 4$, $g'(x) = -4(x+2)^{-5}$. And the interval we are examining is $[-4, 0]$, so $a = -4$ and $b = 0$. This also gives: $g(a) = g(0) = \dfrac{1}{2^4} = \dfrac{1}{16}$ and $g(b) = g(-4) = \dfrac{1}{(-2)^4} = \dfrac{1}{16}$

Plugging in to the MVT conclusion:

$$\begin{aligned} -4(c+2)^{-5} &= \frac{\frac{1}{16} - \frac{1}{16}}{0 - (-4)} \\ \frac{-4}{(c+2)^5} &= 0 \end{aligned}$$

Because this is a quotient with a nonzero numerator, there are no values of c that would satisfy this equation. Therefore, there is no value of c in $[-4, 0]$ such that MVT holds. This is not a contradiction of the theorem because the function $g(x)$ is not continuous on $[-4, 0]$ because there is a vertical asymptote when $x = -2$, which lies within the interval in question.