1. [2360/062725 (18 pts)] Consider the vectors $\left\{ \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} \right\}$ in $\mathbb{R}^3$.

   (a) (6 pts) Do the vectors form a basis for $\mathbb{R}^3$. Why or why not?

   (b) (6 pts) Find all solutions of $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{0}}$ where the columns of $\mathbf{A}$ are the vectors in the set in the order shown.

   (c) (6 pts) Based on your answer in (b), find the dimension of and a basis for the subspace of $\mathbb{R}^3$ consisting of all solutions of the equation $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{0}}$.

SOLUTION:

(a) We need to see if the only solution to the equation

$$c_1 \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix} + c_3 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

is $c_1 = c_2 = c_3 = 0$. This is equivalent to

$$\begin{bmatrix} 1 & 5 & -3 \\ -1 & -4 & 1 \\ -2 & -7 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

with augmented matrix

$$\left[\begin{array}{rrr|r} 1 & 5 & -3 & 0 \\ -1 & -4 & 1 & 0 \\ -2 & -7 & 0 & 0 \end{array} \right] \xrightarrow{\text{RREF}} \left[\begin{array}{rrr|r} 1 & 0 & 7 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

This matrix has only two pivot columns, implying that the linear system has nontrivial solutions, further implying that the vectors are linearly dependent. Consequently, even though we have three vectors in a vector space of dimension 3, since they are not linearly independent, they cannot form a basis for $\mathbb{R}^3$.

Alternatively, the determinant of the coefficient matrix vanishes, meaning that the homogeneous system has nontrivial solutions, implying that the set of vectors is linearly dependent and thus cannot form a basis for $\mathbb{R}^3$.

(b) The given linear system is the same linear system that was used in part (a). From the RREF, the free variable is x_3 which we can set to t, a real number. We then have $x_1 = -7t$ and $x_2 = 2t$ so the solutions to the linear system are $\vec{\mathbf{x}} = t \begin{bmatrix} -7 \\ 2 \\ 1 \end{bmatrix}$, $t \in \mathbb{R}$.

(c) A basis for the solution space of the linear system is $\left\{ \begin{bmatrix} -7 \\ 2 \\ 1 \end{bmatrix} \right\}$ which has dimension 1.

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2. [2360/062725 (10 pts)] Consider the linear system

$$\begin{aligned} x_1 + 3x_3 &= 1 \\ 2x_1 + x_2 + 4x_3 - x_4 &= 2 \\ 3x_1 + 2x_2 + 4x_3 &= -1 \\ 3x_2 - x_3 &= -2 \end{aligned}$$

Use Cramer's Rule to find x_2. No points will be awarded for using any other technique.

SOLUTION:

In matrix form $(\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}})$ this system is

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 2 & 1 & 4 & -1 \\ 3 & 2 & 4 & 0 \\ 0 & 3 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ -2 \end{bmatrix}$$

The determinant of the coefficient matrix is as follows, which we compute by expanding along column 4 (then expanding along the first row in the resulting 3×3 matrix)

$$\begin{vmatrix} 1 & 0 & 3 & 0 \\ 2 & 1 & 4 & -1 \\ 3 & 2 & 4 & 0 \\ 0 & 3 & -1 & 0 \end{vmatrix} = -1(-1)^{2+4} \begin{vmatrix} 1 & 0 & 3 \\ 3 & 2 & 4 \\ 0 & 3 & -1 \end{vmatrix} = -1 \left[1(-1)^{1+1} \begin{vmatrix} 2 & 4 \\ 3 & -1 \end{vmatrix} + 3(-1)^{1+3} \begin{vmatrix} 3 & 2 \\ 0 & 3 \end{vmatrix} \right] = -13$$

We then need to compute $|\mathbf{A}_2|$, given by replacing column two in the coefficient matrix with the right hand side of the system, again expanding along the fourth column (then expanding along the third row in the resulting 3×3 matrix)

$$\begin{vmatrix} 1 & 1 & 3 & 0 \\ 2 & 2 & 4 & -1 \\ 3 & -1 & 4 & 0 \\ 0 & -2 & -1 & 0 \end{vmatrix} = -1(-1)^{2+4} \begin{vmatrix} 1 & 1 & 3 \\ 3 & -1 & 4 \\ 0 & -2 & -1 \end{vmatrix} = -1 \left[-2(-1)^{3+2} \begin{vmatrix} 1 & 3 \\ 3 & 4 \end{vmatrix} + (-1)(-1)^{3+3} \begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} \right] = 6$$

Thus, Cramer's Rule gives $x_2 = |\mathbf{A}_2|/|\mathbf{A}| = -6/13$.

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3. [2360/062725 (10 pts)] Given that $\mathbb{M}_{22}$ is the vector space of all 2×2 matrices, determine if the following subsets, $\mathbb{W}$, are subspaces of $\mathbb{M}_{22}$. Justify your answers.

   (a) (5 pts) $\mathbb{W}$ is the set of matrices of the form $\begin{bmatrix} a & -b \\ b & c \end{bmatrix}$ where a, b, c are real numbers.

   (b) (5 pts) $\mathbb{W}$ is the set of matrices of the form $\begin{bmatrix} 2 & a \\ -a & 3 \end{bmatrix}$ where a is a real number.

SOLUTION:

(a) The zero vector, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, is in $\mathbb{W}$ so we check for closure using linear combinations. Let

$$\begin{bmatrix} a_1 & -b_1 \\ b_1 & c_1 \end{bmatrix}, \begin{bmatrix} a_2 & -b_2 \\ b_2 & c_2 \end{bmatrix} \in \mathbb{W} \text{ and } p, q \in \mathbb{R}$$

Then

$$p \begin{bmatrix} a_1 & -b_1 \\ b_1 & c_1 \end{bmatrix} + q \begin{bmatrix} a_2 & -b_2 \\ b_2 & c_2 \end{bmatrix} = \begin{bmatrix} pa_1 + qa_2 & -(pb_1 + qb_2) \\ pb_1 + qb_2 & pc_1 + qc_2 \end{bmatrix} \in \mathbb{W}$$

showing that the $\mathbb{W}$ is closed under vector addition and scalar multiplication and is thus a subspace of $\mathbb{M}_{22}$.

(b) Since the zero vector is not in the form of the given matrix, it is not in $\mathbb{W}$, therefore $\mathbb{W}$ is not a subspace.

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4. [2360/062725 (18 pts)] If $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$, calculate the following, if possible. If not possible, simply write "not possible". Hint: for part (f), consider using properties of determinants.

   (a) $\mathbf{AB}$ (b) $\mathbf{B}^{\mathrm{T}}\mathbf{A}$ (c) $\mathbf{AA}^{-1}$ (d) $\left|\mathbf{A}^{\mathrm{T}}\mathbf{A}\right|$ (e) $(\mathbf{BA})^{\mathrm{T}}$ (f) $\left|\mathbf{BB}^{\mathrm{T}}\mathbf{B}^{-1}\right|$

SOLUTION:

(a) $\mathbf{AB} = \begin{bmatrix} 3 & 2 \\ 7 & 4 \\ 1 & 0 \end{bmatrix}$

(b) not possible

(c) not possible

(d) $\begin{vmatrix} 5 & 2 \\ 2 & 2 \end{vmatrix} = 6$

(e) not possible

(f) $\left|\mathbf{B}\mathbf{B}^{\mathrm{T}}\mathbf{B}^{-1}\right| = |\mathbf{B}|\left|\mathbf{B}^{\mathrm{T}}\right|\left|\mathbf{B}^{-1}\right| = |\mathbf{B}||\mathbf{B}|\left(\frac{1}{|\mathbf{B}|}\right) = |\mathbf{B}| = -2$

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5. [2360/062725 (12 pts)] You are given the matrices $\mathbf{C}, \mathbf{D}$ and column vector $\vec{\mathbf{u}}$ as follows:

$$\mathbf{C} = \begin{bmatrix} 2 & 1 & 2 \\ 4 & 2 & 3 \\ 0 & -1 & 1 \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} -\frac{5}{2} & \frac{3}{2} & \frac{1}{2} \\ 2 & -1 & -1 \\ 2 & -1 & 0 \end{bmatrix} \qquad \vec{\mathbf{u}} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

(a) (5 pts) Compute $\mathbf{DC}$. Be sure to check your answer carefully.

(b) (7 pts) Without performing any elementary row operations or Gauss-Jordan Elimination, and applying what you found in part (a), find the solution of $\mathbf{C}\vec{\mathbf{x}} = \vec{\mathbf{u}}$. No points awarded if directions are not followed.

SOLUTION:

(a) $\mathbf{DC} = \mathbf{I}$ so that $\mathbf{D} = \mathbf{C}^{-1}$

(b) Multiply both sides of the matrix equation by $\mathbf{D}$ to yield

$$\begin{aligned} \mathbf{D}\left(\mathbf{C}\vec{\mathbf{x}}\right) &= \mathbf{D}\vec{\mathbf{u}} \\ \mathbf{C}^{-1}\left(\mathbf{C}\vec{\mathbf{x}}\right) &= \mathbf{D}\vec{\mathbf{u}} \\ \left(\mathbf{C}^{-1}\mathbf{C}\right)\vec{\mathbf{x}} &= \mathbf{D}\vec{\mathbf{u}} \\ \mathbf{I}\vec{\mathbf{x}} &= \mathbf{D}\vec{\mathbf{u}} \\ \vec{\mathbf{x}} &= \begin{bmatrix} -\frac{5}{2} & \frac{3}{2} & \frac{1}{2} \\ 2 & -1 & -1 \\ 2 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \end{aligned}$$

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6. [2360/062725 (12 pts)] The augmented matrix of the linear system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has been transformed, through a number of elementary row operations, to the following:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & k & 1 & 2 \\ 0 & 0 & k-1 & k^2-1 \end{array}\right]$$

where k is a real number parameter. For which value(s) of k, if any, does the system have ...

(a) (4 pts) exactly one solution?

(b) (4 pts) no solution?

(c) (4 pts) infinitely many solutions?

SOLUTION:

Begin by noting that the determinant of the coefficient matrix is $k(k-1)$.

(a) $k \neq 0, 1$. In this case the determinant is nonzero, implying a unique solution exists.

(b) $k = 0$. In this case the matrix becomes

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -1 & -1 \end{array}\right] \xrightarrow{R_3^* = R_2 + R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

(c) $k = 1$. In this case we have

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[\text{RREF}]{R_1^* = -1R_2 + R_1} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

implying the existence of a free parameter and infinitely many solutions.

7. [2360/062725 (10 pts)] Determine if the set of vectors $\{1, 1-t, 2-4t+t^2, 6-18t+9t^2-t^3\}$ forms a basis for $\mathbb{P}_3$. Be sure to provide correct justification.

SOLUTION:

Alternative 1:

$$W\left[1, 1-t, 2-4t+t^2, 6-18t+9t^2-t^3\right](t) = \begin{vmatrix} 1 & 1-t & 2-4t+t^2 & 6-18t+9t^2-t^3 \\ 0 & -1 & -4+2t & -18+18t-3t^2 \\ 0 & 0 & 2 & 18-6t \\ 0 & 0 & 0 & -6 \end{vmatrix} = 12 \neq 0$$

The nonvanishing of the Wronskian implies the functions are linearly independent. Since we have four linearly independent functions in a vector space of dimension 4, these vectors form a basis for $\mathbb{P}_3$.

Alternative 2:

Let $a_0 + a_1 t + a_2 t^2 + a_3 t^3$ be an arbitrary vector in $\mathbb{P}_3$ where $a_0, a_1, a_2, a_3 \in \mathbb{R}$. Can we can find constants c_1, c_2, c_3, c_4 such that

$$c_1(1) + c_2(1-t) + c_3(2-4t+t^2) + c_4(6-18t+9t^2-t^3) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

Equating coefficients of the various powers of t on both sides of the previous equation yields the linear system

$$\begin{bmatrix} 1 & 1 & 2 & 6 \\ 0 & -1 & -4 & -18 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

The determinant of the coefficient matrix is 1 implying that the system has a unique solution. This implies that the vectors span $\mathbb{P}_3$. Furthermore, if the right hand side of the above linear system is replaced with $\vec{\mathbf{0}}$, the resulting system has only the trivial solution, showing that the vectors are linearly independent. We therefore have a linearly independent spanning set of vectors, in other words the vectors form a basis for $\mathbb{P}_3$.

8. [2360/062725 (10 pts)] Let $\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 1 & 1 & 0 \end{bmatrix}$.

(a) (4 pts) Find all the eigenvalues of $\mathbf{A}$ and state the multiplicity of each.

(b) (6 pts) For the eigenvalue with algebraic multiplicity greater than 1, find its geometric multiplicity and a basis for its eigenspace.

SOLUTION:

(a) The characteristic equation gives

$$|\mathbf{A} - \lambda \mathbf{I}| = \begin{bmatrix} 1-\lambda & 0 & 2 \\ 0 & 1-\lambda & -2 \\ 1 & 1 & -\lambda \end{bmatrix} = -\lambda(1-\lambda)^2 = 0 \implies \lambda = 0 \,(\text{algebraic multiplicity } 1); \lambda = 1 \,(\text{algebraic multiplicity } 2)$$

(b) Solve the linear system of algebraic equations $(\mathbf{A} - 1\mathbf{I})\,\vec{\mathbf{v}} = \vec{\mathbf{0}}$ with augmented matrix

$$\left[\begin{array}{ccc|c} 0 & 0 & 2 & 0 \\ 0 & 0 & -2 & 0 \\ 1 & 1 & -1 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

which has solution $\vec{\mathbf{v}} = \begin{bmatrix} -t \\ t \\ 0 \end{bmatrix}$, $t \in \mathbb{R}$. A basis for the eigenspace is therefore $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$ having dimension 1.