Department of Applied Mathematics
Preliminary Examination in Numerical Analysis
May 2025

Instructions

You have three hours to complete this exam. Submit solutions to four (and no more) of the following six problems. Please start each problem on a new page. You MUST prove your conclusions or show a counter-example for all problems unless otherwise noted. Write your student ID number (not your name!) on your exam.

Problem 1: Rootfinding methods
Suppose we want to compute the square root $x = \sqrt{a}$ for some $a > 0$. We can do this by using Newton's method to solve $x^2 = a$.

(a) Set up the problem outlined above as a rootfinding problem for a function $f(x)$. Suppose that α is a root of $f(x)$ and that the initial guess x_0 is sufficiently close to α. What are sufficient conditions required for local convergence of Newton's method? Check that this problem satisfies those requirements.

(b) Beginning with some $x_0 > 0$, the iteration formula for this is

$$x_{k+1} = \frac{1}{2}\left(x_k + \frac{a}{x_k}\right).$$

Derive this iteration formula.

(c) Suppose that the conditions in part (a) are satisfied. Show that Newton's method converges locally for x_0 sufficiently close to α.

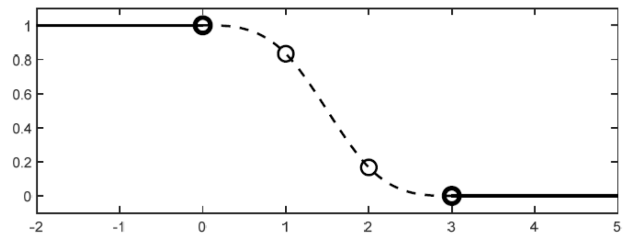
(d) Show that the sequence of iterations is locally quadratically convergent. (Solving this part (d) correctly also gives full credit for part (c).)

Problem: Interpolation and Approximation
We consider in this problem cubic splines.

(a) Determine how many extra end conditions are needed to determine a cubic spline uniquely. Also, describe two common ways to ensure this uniqueness.

(b) Define what is meant by a B-spline (also known as a basis spline).

(c) The cubic spline that transitions the fastest from identically one to identically zero on a unit-spaced grid looks graphically as follows:



Determine the exact function values for this spline at the two internal node points (located at $x = 1$ and $x = 2$).

Problem 3: Numerical integration

Gaussian quadrature is commonly used for numerical approximation of integrals. One generalization is to instead apply it to approximate infinite sums. Determine the nodes x_1, x_2 and weights w_1, w_2 so that the formula

$$\sum_{n=0}^{\infty} \frac{f(n)}{n!} = w_1 f(x_1) + w_2 f(x_2)$$

becomes exact for polynomials $f(x)$ of as high degree as possible.

Hint: The inner product to use becomes

$$< f, g >= \sum_{n=0}^{\infty} \frac{1}{n!} f(n) g(n)$$

Sums of the form $\sum_{n=0}^{\infty} \frac{n^p}{n!}$ can be found in closed form by considering the derivative of $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ at $x = 1$, multiplying by x and differentiating, etc.

Problem 4: Numerical Linear Algebra
Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be a diagonalizable matrix with eigenvalues

$$|\lambda_1| > |\lambda_2| \geq \cdots \geq |\lambda_n| > 0,$$

and corresponding linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, such that

$$\mathbf{A}\mathbf{v}_i = \lambda_i \mathbf{v}_i, \quad \text{for } i = 1, 2, \ldots, n.$$

(a) State the condition(s) required for convergence of the Power Method to the dominant eigenvector. Write pseudocode for the Power Method to approximate the eigenvector associated with the largest magnitude eigenvalue of $\mathbf{A}$.

(b) Assume that the condition(s) in part (a) are satisfied. Show that the Power Method converges to the dominant eigenvector. (For simplicity, you can assume that $\lambda_1 > 0$.)

(c) Using your solution in part (b), show that the Rayleigh Quotient converges to the dominant eigenvalue.

(d) Suppose $|\lambda_{n-1}| > |\lambda_n|$. Modify the Power Method to approximate an eigenvector corresponding to the smallest magnitude eigenvalue. Explain why your modification works. State the condition(s) required for convergence and write pseudocode for this procedure.

Problem 5: Numerical ODE (25 points)

We wish to solve an IVP for a system of N 1st order ODEs $\{\vec{y}'(t) = f(t, \vec{y}),\ \vec{y}(0) = \vec{y}_0\}$. We consider four linear, multistep methods (LMMs) derived from finite difference formulas:

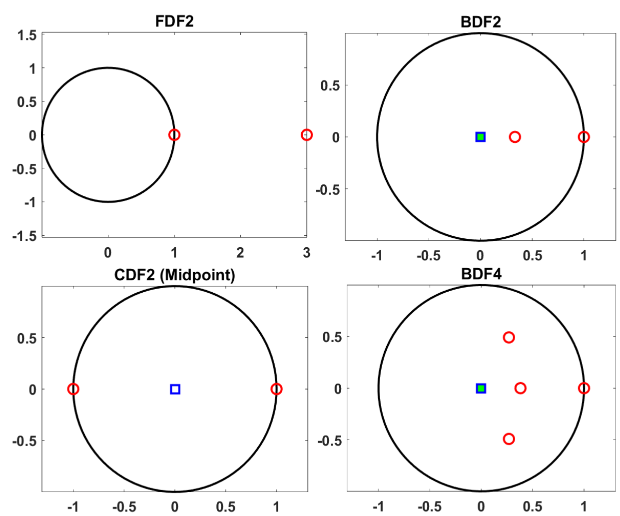
$$\begin{aligned}
\text{FDF2}: &\ \frac{1}{2}(-3y_n + 4y_{n+1} - y_{n+2}) = \Delta t f(t_n, y_n) \\
\text{CDF2}: &\ \frac{1}{2}(-y_n + y_{n+2}) = \Delta t f(t_{n+1}, y_{n+1}) \\
\text{BDF2}: &\ \frac{1}{2}(y_n - 4y_{n+1} + 3y_{n+2}) = \Delta t f(t_{n+2}, y_{n+2}) \\
\text{BDF4}: &\ \frac{1}{12}(3y_n - 16y_{n+1} + 36y_{n+2} - 48y_{n+3} + 25y_{n+4}) = \Delta t f(t_{n+4}, y_{n+4})
\end{aligned}$$

(a) Write down a definition of the **truncation error** for the FDF2 multistep method. This method is obtained from a formula for the derivative of a smooth function $Y(t)$, satisfying:

$$Y'(t) - \frac{-3Y(t) + 4Y(t+\Delta t) - Y(t+2\Delta t)}{2\Delta t} = O(\Delta t^2)$$

What can you conclude about the truncation errors for FDF2 from this? If FDF2 converges, what is expected for the global error?

(b) Recall that for LMM $\sum_{m=0}^{p} a_m y_{n+m} = \Delta t \sum_{m=0}^{p} b_m f(t_{n+m}, y_{n+m})$ we can use $\rho(w) = \sum_{m=0}^{p} a_m w^m$ and $\sigma(w) = \sum_{m=0}^{p} b_m w^m$ to analyze its properties.



In each of these plots, roots of $\rho(w)$ are denoted as circles, and roots of $\sigma(w)$ as squares. If the root is simple, the marker face is empty. If it is repeated, it is filled with color.

Assume all of the methods presented above are consistent. Using the information in these plots or otherwise, indicate what you know about each method's (i) stability, (ii) convergence and (iii) relative (strong) stability. Be sure to justify each of these conclusions.

(c) Write down a simple pseudocode for a function using BDF2 to integrate the IVP for the system of N ODEs from time 0 to time T taking n_t time steps. Be sure to indicate inputs and outputs required for this routine.

Problem 6: Numerical PDE (25 points)

Consider the following elliptic two-point Boundary Value Problem (BVP) on $(0, 1)$:

$$-au_{xx}(x) = -k^2 u(x) + f(x) \quad x \in (0,1)$$
$$u(0) = g_0 \quad u(1) = g_1$$

with $a > 0, k \geq 0$, $g_0, g_1 \in \mathbb{R}$ and f smooth.

(a) For a regular grid of points $x_i = ih$, with $i = 0, 1, \ldots, n$ and spacing $h = 1/n$, we denote our approximation $U_i \simeq u(x_i)$. Using second differences for u_{xx}, write down a system of equations for each unknown U_i. Indicate how the boundary data is incorporated.

(b) Note that by collecting all unknown entries in a vector $\boldsymbol{U}$, we can write the equations above as a linear system of the form $\mathsf{M}\boldsymbol{U} = \boldsymbol{b}$. Describe the entries and structure of matrix M. If possible, provide an estimate for its condition number $\kappa(\mathsf{M})$.

(c) Consider the associated parabolic problem:

$$u_t(x,t) = au_{xx}(x) - k^2 u(x) + f(x) \quad x \in (0,1), t \in (0,T)$$
$$u(0,t) = g_0 \quad u(0,t) = g_1, t \in (0,T)$$
$$u(x,0) = \phi(x) \quad x \in (0,1)$$

for ϕ smooth. Use the same discretization in space as in (a)-(b) to define a linear IVP on the vector $\boldsymbol{U}(t)$ with entries $U_i(t) \simeq u(x_i, t)$. Then, use a Backward Euler method in time to write down a time-stepping formula for $\boldsymbol{U}(t_{k+1})$ in terms of $\boldsymbol{U}(t_k)$, for $t_k = k\Delta t$, $\Delta t = T/n_t$.

(d) Assume M is Symmetric Positive Definite (SPD), with eigenvalues in the interval (η_1, η_2) where $\eta_1, \eta_2 \geq 0$, $\eta_1 \leq \lambda_{min}(\mathsf{M})$ and $\eta_2 \geq \lambda_{max}(\mathsf{M})$. Explain how you can use this information to perform a stability analysis on the scheme proposed in (c). Indicate what restriction (if any) there is on timestep size Δt.