Applied Analysis Preliminary Exam (Hints/solutions)
9:00 AM – 12:00 PM, Thursday May 22, 2025

Instructions You have three hours to complete this exam. Work all five problems; there are no optional problems. Each problem is worth 20 points. Please start each problem on a new page. Please clearly indicate any work that you do not wish to be graded (e.g., write SCRATCH at the top of such a page). You MUST prove your conclusions or show a counter-example for all problems unless otherwise noted. In your proofs, you may use any major theorem on the syllabus or discussed in class, unless you are directly proving such a theorem (when in doubt, ask the proctor). If you cannot finish part of a question, you may wish to move on to the next part; problems are graded with partial credit. **Write your student number on your exam, not your name**.

Problem 1 (20 points)

(a) [10] Let p_k, $k \in \mathbb{N}, k \geq 1$ be a sequence of seminorms defined on a real vector space X. In other words, p_k satisfies:

(P1) $p_k(x) \geq 0$ for all $x \in X$;

(P2) $p_k(cx) = |c| p_k(x)$ for all $x \in X$ and $c \in \mathbb{R}$;

(P3) $p_k(x+y) \leq p_k(x) + p_k(y)$;

for all $k \in \mathbb{N}, k \geq 1$. Assume that for every $x \in X$ with $x \neq 0$ there exists at least one index k such that $p_k(x) > 0$ (we say that this sequence is **separating**). Define

$$d(x,y) = \sum_{k=1}^{\infty} 2^{-k} \frac{p_k(x-y)}{1+p_k(x-y)}. \tag{1}$$

prove that d defines a distance on X.

(b) [10] Let $\Omega \subset \mathbb{R}^n$ be an open set with boundary $\partial\Omega$. Let $C(\Omega)$ be the space of all continuous functions on Ω. For every $k \geq 1$ consider the compact subset:

$$A_k = \left\{ x \in \Omega : \|x\| \leq k,\ B(x, 1/k) \subset \Omega \right\}, \tag{2}$$

where $B(x,1/k) \stackrel{\text{def}}{=} \left\{ z \in \mathbb{R}^n, \|x-z\| < 1/k \right\}$, and $\|\cdot\|$ is the Euclidean norm. Define the seminorm on $C(\Omega)$,

$$p_k(f) = \max_{x \in A_k} |f(x)| \tag{3}$$

and d the corresponding metric defined by (1). Prove that the space $C(\Omega)$ equipped with the metric d is a complete metric space.

Solution:

(a) From (1) we see that $d(x,x) = 0$ follows from (P2) by taking $c = 0$ for all k. Taking $c = -1$ in $(P2)$ for all k, we obtain that $d(x,y) = d(y,x)$. To prove the triangle inequality, note that if $c \leq a+b$ then it holds that

$$\frac{c}{1+c} \leq \frac{a+b}{1+a+b}$$

as the function $g(s) = s/(1+s)$ is increasing. Moreover,

$$\frac{a+b}{1+a+b} \leq \frac{a}{1+a} + \frac{b}{1+b}$$

because g is concave down on $(-1, \infty)$. Let $c = p_k(x-z)$, $a = p_k(x-y)$, and $b = p_k(y-z)$ and consider:

$$\begin{aligned} d(x,z) &= \sum_{k=1}^{\infty} 2^{-k} \frac{p_k(x-z)}{1+p_k(x-z)} \\ &\leq \sum_{k=1}^{\infty} 2^{-k} \left(\frac{p_k(x-y)}{1+p_k(x-y)} + \frac{p_k(y-z)}{1+p_k(y-z)} \right) \\ &= d(x,y) + d(y,z). \end{aligned}$$

(b) By problem (a), we now know that $C(\Omega)$ equipped with d is a metric space and so we only need to show completeness. For this purpose, consider a Cauchy sequence $\{f_j\} \subset C(\Omega)$. Thus, for any $\epsilon > 0$ there exists a K_ϵ such that $d(f_i, f_j) < \epsilon$, for all $i, j > K_\epsilon$. Thus, for all k (in the definition of d) we have that

$$\limsup_{i,j\to\infty} p_k(f_i - f_j) = \limsup_{i,j\to\infty} \sup_{x\in A_k} |f_i(x) - f_j(x)| = 0. \tag{4}$$

We have $A_k \uparrow \Omega$, so any point $x \in \Omega$ belongs in a set A_k for some k and thus we conclude that the sequence $\{f_j(x)\}$ is Cauchy and hence converges to some limit point $f(x)$. In fact, every compact subset of Ω is contained in some subset A_k, hence by (4) the convergence $f_j \to f$ must be uniform on compact subsets of Ω. This implies that the limiting function f is a continuous function.

We are left to prove that $\lim_{j\to\infty} d(f_j, f) \to 0$. For this purpose, fix $m \geq 1$ and consider that

$$\begin{aligned} \limsup_{j\to\infty} d(f_j, f) &\leq \limsup_{j\to\infty} \sum_{k=1}^{m} 2^{-k} \frac{p_k(f_j - f)}{1+p_k(f_j - f)} + \limsup_{j\to\infty} \sum_{k=m+1}^{\infty} 2^{-k} \frac{p_k(f_j - f)}{1+p_k(f_j - f)} \\ &\leq 0 + 2^{-m}. \end{aligned}$$

Taking $m \to \infty$ gives the result.

Problem 2 (20 points)

(a) [4] Let $f : X \to X$ be a contraction on a complete metric space and $Y \subset X$ be a closed subset such that $f(Y) \subset Y$. Prove that the unique fixed point of f is in Y.

(b) [5] Let the contraction constant of f be c and its fixed point a. Show that any $x_0 \in X$ and f-iterates $\{x_n\}$ satisfy:

$$d(x_n, a) \leq \frac{c}{1-c} d(x_{n-1}, x_n).$$

(c) [11] Let X be a complete metric space and $f : X \to X$ be a mapping such that some iterate $f^N : X \to X$ is a contraction. Prove that f has a unique fixed point. Moreover, show that the fixed point of f can be obtained by iteration of f starting from any $x_0 \in X$.

Solution:

(a) Since Y is a closed subset of a complete metric space, it is also complete. Thus, we can apply the contraction mapping theorem to $f : Y \to Y$ and we get a unique fixed point in Y.

(b) Let c be the contraction constant. Since a is the fixed point and f is a contraction we see that

$$\begin{aligned} d(x_n, a) &= d(f(x_{n-1}), f(a)) \\ &\leq c d(x_{n-1}, a) \\ &\leq c[d(x_{n-1}, x_n) + d(x_n, a)]. \end{aligned}$$

Solving for $d(x_n, a)$ gives the desired inequality.

(c) Applying the contraction mapping theorem to the function f^N gives a unique fixed point, a, in X. If f has a fixed point b, note that $f^N(b) = b$ and so by uniqueness $b \equiv a$. Moreover, $f(a)$ is also a fixed point of f^N as $f^N(f(a)) = f(f^N(a)) = f(a)$ and thus we have that $f(a) = a$.

Next, let $x_0 \in X$ be arbitrarily chosen. We will show that $\lim_{n\to\infty} f^n(x_0) = a$. For this purpose, let $0 \le r \le N-1$ and consider:

$$f^{Nk+r}(x_0) = f^{Nk}(f^r(x_0)) = (f^N)^k(f^r(x_0)).$$

Let $y_0 = f^r(x_0)$. Since f^N is a contraction, these iterates must converge to a as $k \to \infty$. This limit is independent of the value r such that $0 \le r \le N-1$. Thus, all sequences $\left\{f^{Nk+r}(x_0)\right\}_k$ tend to a as $k \to \infty$. Thus, we have that

$$\lim_{n\to\infty} f^n(x_0) = a$$

independent of x_0.

Problem 3 (20 points) Let K be a real-valued function continuous on $[0,1]\times[0,1]$. We define the linear operator, $T : (\mathscr{C}\big([0,1]\big), \|\ \|_\infty) \to (\mathscr{C}\big([0,1]\big), \|\ \|_\infty)$, such that

$$\forall f \in \mathscr{C}\big([0,1]\big),\ \ \forall x \in [0,1],\ \ [Tf](x) = \int_0^x K(x,t)f(t)dt. \tag{5}$$

(a) [2] Prove that T is a bounded operator from $(\mathscr{C}\big([0,1]\big), \|\ \|_\infty)$ to $(\mathscr{C}\big([0,1]\big), \|\ \|_\infty)$.

(b) [6] Prove that T is compact.

(c) [12] Compute the spectral radius of T, and find $\sigma(T)$, the spectrum of T. Hint: you may prove the following intermediate estimate, $\forall n \in \mathbb{N}, n \ge 1, \forall x \in [0,1]$, $\big|[T^n f](x)\big| \le \|f\|_\infty \big[x\|K\|_\infty\big]^n/n!$

Solution:

(a) A quick calculation shows that

$$\forall x,y \in [0,1], \quad \big|[Tf](x) - [Tf](y)\big| \le \Big(\|K\|_\infty|x-y| + \sup_{t\in[0,1]} |K(x,t) - K(y,t)|\Big)\|f\|_\infty \tag{6}$$

The function K is continuous on $[0,1]\times[0,1]$ so it is uniformly continuous, and therefore $\forall\varepsilon > 0, \exists\delta > 0$ such that

$$\forall x,y,t \in [0,1]\times[0,1], \qquad \big(|x-y| < \delta \Rightarrow |K(x,t) - K(y,t)| < \varepsilon/2\big). \tag{7}$$

Now take $\delta' = \min\big(\delta, \varepsilon/(2\|K\|_\infty)\big)$, we have

$$\forall x,y, \quad |x-y| < \delta' \Rightarrow \big|[Tf](x) - [Tf](y)\big| \le \big(\varepsilon/2 + \delta'\|K\|_\infty\big)\|f\|_\infty \le \varepsilon\|f\|_\infty \tag{8}$$

We conclude that Tf is (uniformly) continuous; in fact we have $T\big(B_{\mathscr{C}\left([0,1]\right)}(0,1)\big)$ is equicontinuous. We will need this later. T is obviously linear. Also, we have $\|Tf\|_\infty \le \|K\|_\infty\|f\|_\infty$, so $T \in \mathcal{B}(\mathscr{C}\big([0,1]\big))$.

(b) Let $\mathbb{B} \stackrel{\text{def}}{=} B_{\mathscr{C}\left([0,1]\right)}(0,1)$ be the unit ball in $\mathscr{C}\big([0,1]\big)$. We use the Ascoli & Arzelà theorem to prove that $T(\mathbb{B})$ has compact closure. This involves proving that

- $T(\mathbb{B})$ is equicontinuous, which was proved in the previous question;
- $T(\mathbb{B})$ is uniformly bounded in $\mathscr{C}\big([0,1]\big)$: we have $f \in \mathbb{B} \Rightarrow \|Tf\|_\infty \le \|K\|_\infty$.

(c) We can prove by induction that

$$\forall n \in \mathbb{N}, n \ge 1, \forall x \in [0,1], \quad \big|[T^n f](x)\big| \le \|f\|_\infty \|K\|_\infty^n \frac{x^n}{n!} \tag{9}$$

whence

$$\forall n \in \mathbb{N}, n \ge 1, \|T^n\| \le \frac{\big[\|K\|_\infty\big]^n}{n!}. \tag{10}$$

We conclude that $r(T) = \lim_{n\to\infty} \|T^n\|^{1/n} = 0$. We have $\sigma(T) \subset B(0, r(T)) = \{0\}$. But T is compact so 0 is always part of the spectrum. We conclude $\sigma(T) = \{0\}$.

Problem 4 (20 points)

(a) [8] Let $\mathcal{H}$ be a Hilbert space and let $T : \mathcal{H} \to \mathcal{H}$ be a linear map. Prove that

$$T \text{ is selfadjoint} \Longrightarrow T \text{ is continuous.} \tag{11}$$

(b) [2] Let $\mathbb{C}[X]$ be the vector space of polynomials on $[0, 1]$ with complex coefficients, equipped with the inner product

$$\langle P, Q\rangle = \int_0^1 P(t)\overline{Q(t)}dt, \tag{12}$$

and let $\mathcal{H}$ the pre-Hilbert subspace of $\mathbb{C}[X]$ defined by $\mathcal{H} = \Big\{P \in \mathbb{C}[X], \quad P(0) = P(1) = 0\Big\}$. We consider the linear map T defined by

$$T : \mathcal{H} \longrightarrow \mathbb{C}[X], \tag{13}$$

$$P \longmapsto iP', \tag{14}$$

where P' is the derivative of P. Prove that

$$\forall P, Q \in \mathcal{H}, \quad \langle T(P), Q\rangle = \langle P, T(Q)\rangle. \tag{15}$$

(c) [10] Prove that T is not continuous, and resolve the apparent contradiction with (11).

Solution:

(a) First proof (from a student). We use the closed graph theorem. Let $\{x_n\} \subset \mathcal{H}$, such that $\lim_{n\to\infty} x_n = 0$. We prove $\lim_{n\to\infty} Tx_n = 0$. We have $\forall y \in \mathcal{H}, \langle Tx_n, y\rangle = \langle x_n, Ty\rangle$. Then by continuity of the inner product, $\lim_{n\to\infty} \langle x_n, Ty\rangle = 0$. We conclude that $\forall y \in \mathcal{H}, \lim_{n\to\infty} \langle Tx_n, y\rangle = 0$, whence $\lim_{n\to} Tx_n = 0$.

Second proof (more elaborate). We first prove a small result. Let $T : \mathcal{H} \to \mathcal{H}$ be a linear map. Then,

$$T \quad \text{is continuous} \iff \Big[\forall \varphi \in \mathcal{H}', \varphi \circ T \in \mathcal{H}'\Big] \tag{16}$$

where $\mathcal{H}'$ is the topological dual of $\mathcal{H}$. The reverse direction is the only interesting direction. The closed graph theorem allows us to conclude using a standard elementary proof (left as an exercise).
We now prove (11). Let $\varphi \in \mathcal{H}'$, we need to prove that $\varphi \circ T \in \mathcal{H}'$. We use Riesz's representation theorem to find $u \in \mathcal{H}$ such that $\varphi(x) = \langle x, u\rangle$. Therefore $\varphi(Tx) = \langle Tx, u\rangle = \langle x, Tu\rangle$ by assumption, and we conclude that $\varphi \circ T \in \mathcal{H}'$ so T is continuous.

(b) Integration by part yields the result.

(c) We define the following sequence of polynomials, $P_n(t) \stackrel{\text{def}}{=} t^n(1-t)$. An elementary calculation yields

$$\|P_n\| = \frac{\sqrt{2}}{\sqrt{\big((2n+1)(2n+2)(2n+3)\big)}} = \frac{1}{2}n^{-3/2}(1 + o(1)). \tag{17}$$

Also,

$$\|P_n'\| \geq \|t^n\| - n\|P_{n-1}\| \geq \big(\frac{1}{\sqrt{2}} - \frac{1}{2}\big)n^{-1/2}(1 + o(1)). \tag{18}$$

We conclude that

$$\frac{\|P_n'\|}{\|P_n\|} \geq (\sqrt{2} - 1)n(1 + o(1)) \tag{19}$$

and therefore the operator is not bounded. The result (11) does not apply since $\mathcal{H}$ is not complete (it is not even closed). We note in passing that $\mathbb{C}[X]$ is not complete either.

Problem 5 (20 points) Let $(\Omega, \mathscr{A}, \mu)$ be a measure space, such that $\mu(\Omega) < \infty$. Let $f : \Omega \mapsto \mathbb{R}$ be a measurable function. We define

$$\forall n \in \mathbb{N}, \quad A_n = \Big\{\omega \in \Omega; \quad n \leq |f(\omega)|\Big\} \quad \text{and} \quad B_n = \Big\{\omega \in \Omega; \quad n < |f(\omega)| \leq n+1\Big\}. \tag{20}$$

[10 + 10] Prove that

$$\int |f| d\mu < \infty \Leftrightarrow \sum_{n=0}^{\infty} n\, \mu(B_n) < \infty \Leftrightarrow \sum_{n=0}^{\infty} \mu(A_n) < \infty. \tag{21}$$

Solution:

(a) Because f is measurable, the sets $A_n, B_n \in \mathscr{A}$. Also,

$$\forall \omega \in \Omega, \sum_{n \in \mathbb{N}} n \mathbb{1}_{B_n}(\omega) \leq |f(\omega)| \leq \sum_{n \in \mathbb{N}} (n+1) \mathbb{1}_{B_n}(\omega). \tag{22}$$

The monotone convergence theorem (applied to the first and last terms) and the monotonicity of the integral yield

$$\sum_{n \in \mathbb{N}} n\, \mu(B_n) \leq \int_{\Omega} |f| d\mu \leq \sum_{n \in \mathbb{N}} (n+1) \mu(B_n). \tag{23}$$

Whence, $\int_{\Omega} |f| d\mu < \infty \Rightarrow \sum_{n \in \mathbb{N}} n\, \mu(B_n) < \infty$. To prove the converse, we observe that $B_n \cap B_m = \emptyset$ if $n \neq m$. Therefore

$$\mu\big(\bigcup_{n \in \mathbb{N}} B_n\big) = \sum_{n \in \mathbb{N}} \mu(B_n) \leq \mu(\Omega) < \infty. \tag{24}$$

Whence,

$$\sum_{n \in \mathbb{N}} n\, \mu(B_n) < \infty \Rightarrow \sum_{n \in \mathbb{N}} (n+1) \mu(B_n) < \infty \Rightarrow \int_{\Omega} |f| d\mu < \infty. \tag{25}$$

Now, for the second equivalence, we modify B_n ever so slightly and define

$$C_n = \Big\{\omega \in \Omega; \quad n \leq |f(\omega)| < n+1\Big\}. \tag{26}$$

The same proof (mutatis mutandis) yields,

$$\int |f| d\mu < \infty \Leftrightarrow \sum_{n=0}^{\infty} n\, \mu(C_n) < \infty. \tag{27}$$

Now observe that C_n is the thin slice that one can add to A_{n+1} to get A_n. To wit, $A_n = A_{n+1} \cup C_n$ and $A_{n+1} \cap C_n = \emptyset$. Therefore $\mu(C_n) = \mu(A_n) - \mu(A_{n+1})$. A small computation shows

$$\sum_{n=1}^{n} \mu(A_n) = \sum_{n=0}^{N} n\, \mu(C_n) + n\, \mu(A_{N+1}). \tag{28}$$

Thus,

$$\sum_{n=1}^{\infty} \mu(A_n) < \infty \Rightarrow \sum_{n=0}^{\infty} n\, \mu(C_n) < \infty \Rightarrow \int |f| d\mu < \infty. \tag{29}$$

Conversely, if $\int |f| d\mu < \infty$ then $\sum_{n=0}^{\infty} n\, \mu(C_n) < \infty$. But we always have $N \mathbb{1}_{A_{N+1}} \leq |f|$, so $n\, \mu(A_{N+1}) \leq \int |f| d\mu < \infty$, whence

$$\sum_{n=1}^{n} \mu(A_n) = \sum_{n=0}^{N} n\, \mu(C_n) + n\, \mu(A_{N+1}) \leq \sum_{n=0}^{N} n\, \mu(C_n) + \int |f| d\mu, \tag{30}$$

and we conclude that $\sum_{n=1}^{\infty} \mu(A_n) < \infty$.