Applied Analysis Preliminary Exam
9:00 AM – 12:00 PM, Thursday May 22, 2025

Instructions You have three hours to complete this exam. Work all five problems; there are no optional problems. Each problem is worth 20 points. Please start each problem on a new page. Please clearly indicate any work that you do not wish to be graded (e.g., write SCRATCH at the top of such a page). You MUST prove your conclusions or show a counter-example for all problems unless otherwise noted. In your proofs, you may use any major theorem on the syllabus or discussed in class, unless you are directly proving such a theorem (when in doubt, ask the proctor). If you cannot finish part of a question, you may wish to move on to the next part; problems are graded with partial credit. **Write your student number on your exam, not your name**.

Problem 1 (20 points)

(a) [10] Let p_k, $k \in \mathbb{N}, k \geq 1$ be a sequence of seminorms defined on a real vector space X. In other words, p_k satisfies:

(P1) $p_k(x) \geq 0$ for all $x \in X$;

(P2) $p_k(cx) = |c| p_k(x)$ for all $x \in X$ and $c \in \mathbb{R}$;

(P3) $p_k(x+y) \leq p_k(x) + p_k(y)$;

for all $k \in \mathbb{N}, k \geq 1$. Assume that for every $x \in X$ with $x \neq 0$ there exists at least one index k such that $p_k(x) > 0$ (we say that this sequence is **separating**). Define

$$d(x,y) = \sum_{k=1}^{\infty} 2^{-k} \frac{p_k(x-y)}{1 + p_k(x-y)}. \tag{1}$$

prove that d defines a distance on X.

(b) [10] Let $\Omega \subset \mathbb{R}^n$ be an open set with boundary $\partial\Omega$. Let $C(\Omega)$ be the space of all continuous functions on Ω. For every $k \geq 1$ consider the compact subset:

$$A_k = \left\{ x \in \Omega : \|x\| \leq k,\ B(x, 1/k) \subset \Omega \right\}, \tag{2}$$

where $B(x,1/k) \stackrel{\text{def}}{=} \left\{ z \in \mathbb{R}^n, \|x - z\| < 1/k \right\}$, and $\|\cdot\|$ is the Euclidean norm. Define the seminorm on $C(\Omega)$,

$$p_k(f) = \max_{x \in A_k} |f(x)| \tag{3}$$

and d the corresponding metric defined by (1). Prove that the space $C(\Omega)$ equipped with the metric d is a complete metric space.

Problem 2 (20 points)

(a) [4] Let $f : X \to X$ be a contraction on a complete metric space and $Y \subset X$ be a closed subset such that $f(Y) \subset Y$. Prove that the unique fixed point of f is in Y.

(b) [5] Let the contraction constant of f be c and its fixed point a. Show that any $x_0 \in X$ and f-iterates $\{x_n\}$ satisfy:

$$d(x_n, a) \leq \frac{c}{1-c} d(x_{n-1}, x_n).$$

(c) [11] Let X be a complete metric space and $f : X \to X$ be a mapping such that some iterate $f^N : X \to X$ is a contraction. Prove that f has a unique fixed point. Moreover, show that the fixed point of f can be obtained by iteration of f starting from any $x_0 \in X$.

Problem 3 (20 points) Let K be a real-valued function continuous on $[0,1]\times[0,1]$. We define the linear operator, $T : (\mathscr{C}\big([0,1]\big), \|\ \|_\infty) \to (\mathscr{C}\big([0,1]\big), \|\ \|_\infty)$, such that

$$\forall f \in \mathscr{C}\big([0,1]\big),\ \ \forall x \in [0,1],\ \ [Tf](x) = \int_0^x K(x,t)f(t)dt. \tag{5}$$

(a) [2] Prove that T is a bounded operator from $(\mathscr{C}\big([0,1]\big), \|\ \|_\infty)$ to $(\mathscr{C}\big([0,1]\big), \|\ \|_\infty)$.

(b) [6] Prove that T is compact.

(c) [12] Compute the spectral radius of T, and find $\sigma(T)$, the spectrum of T. Hint: you may prove the following intermediate estimate, $\forall n \in \mathbb{N}, n \geq 1, \forall x \in [0,1]$, $\big|[T^n f](x)\big| \leq \|f\|_\infty \big[x\|K\|_\infty\big]^n/n!$

Problem 4 (20 points)

(a) [8] Let $\mathcal{H}$ be a Hilbert space and let $T : \mathcal{H} \to \mathcal{H}$ be a linear map. Prove that

$$T \text{ is selfadjoint} \Longrightarrow T \text{ is continuous.} \tag{11}$$

(b) [2] Let $\mathbb{C}[X]$ be the vector space of polynomials on $[0, 1]$ with complex coefficients, equipped with the inner product

$$\langle P, Q \rangle = \int_0^1 P(t)\overline{Q(t)}dt, \tag{12}$$

and let $\mathcal{H}$ the pre-Hilbert subspace of $\mathbb{C}[X]$ defined by $\mathcal{H} = \Big\{P \in \mathbb{C}[X], \quad P(0) = P(1) = 0\Big\}$. We consider the linear map T defined by

$$T : \mathcal{H} \longrightarrow \mathbb{C}[X], \tag{13}$$

$$P \longmapsto iP', \tag{14}$$

where P' is the derivative of P. Prove that

$$\forall P, Q \in \mathcal{H}, \quad \langle T(P), Q \rangle = \langle P, T(Q) \rangle. \tag{15}$$

(c) [10] Prove that T is not continuous, and resolve the apparent contradiction with (11).

Problem 5 (20 points) Let $\big(\Omega, \mathscr{A}, \mu\big)$ be a measure space, such that $\mu(\Omega) < \infty$. Let $f : \Omega \mapsto \mathbb{R}$ be a measurable function. We define

$$\forall n \in \mathbb{N}, \quad A_n = \Big\{\omega \in \Omega; \quad n \leq |f(\omega)|\Big\} \quad \text{and} \quad B_n = \Big\{\omega \in \Omega; \quad n < |f(\omega)| \leq n+1\Big\}. \tag{20}$$

[10 + 10] Prove that

$$\int |f| d\mu < \infty \Leftrightarrow \sum_{n=0}^{\infty} n\, \mu(B_n) < \infty \Leftrightarrow \sum_{n=0}^{\infty} \mu(A_n) < \infty. \tag{21}$$