1. For each of the following, provide a brief proof or justification.

   (a) (6 points) Let $\mathbf{x} \in \mathbb{C}^n$. Is $\sqrt{\mathbf{x}^T\mathbf{x}}$ a valid norm?

   (b) (6 points) Let $A = \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}$. Does the system $A\mathbf{x} = \mathbf{b}$ have a unique least squares solution?

   (c) (6 points) Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ all be vectors in the same inner product space. If $\langle \mathbf{u}, \mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle$ is it necessarily true that $\mathbf{u} = \mathbf{v}$?

   (d) (6 points) Let A^+ be the pseudo-inverse of A. Does $A^+AA^+ = A^+$?

   (e) (6 points) Does whether a set of vectors is linearly independent depend on the specific choice of inner product?

   Solutions:

   (a) No, it fails positivity. Take for example $\mathbf{x} = \begin{pmatrix} 0 \\ i \end{pmatrix}$, then $\sqrt{\mathbf{x}^T\mathbf{x}} = \sqrt{-1} = i$ which is not real.

   (b) No, the solution will not be unique since the columns of A are linearly dependent.

   (c) No, as a counterexample, take the standard dot product, $\mathbf{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$, and $\mathbf{w} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then $\mathbf{u}^T\mathbf{w} = \mathbf{v}^T\mathbf{w} = 0$ but $\mathbf{u} \neq \mathbf{v}$.

   (d) Yes. Let A have the reduced SVD $A = P\Sigma Q^T$, then $A^+ = Q\Sigma^{-1}P^T$ and we have

   $$\begin{aligned} A^+AA^+ &= Q\Sigma^{-1}P^TP\Sigma Q^TQ\Sigma^{-1}P^T \\ &= Q\Sigma^{-1}\Sigma\Sigma^{-1}P^T \\ &= Q\Sigma^{-1}P^T \\ &= A^+ \end{aligned}$$

   (e) No, the linear dependence or independence of vectors is determined by the solutions to $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$, independent of the inner product.

2. (20 points) Suppose $A = SJS^{-1}$, the Jordan decomposition of matrix A, and matrix J is given by

$$J = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) List all eigenvalues of A, together with their associated algebraic and geometric multiplicities.

(b) Is A singular? Why or why not?

(c) Is A complete? Why or why not?

(d) If $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_6$ are the columns of S, which of the following, if any, are eigenvectors of A: (i) $\mathbf{s}_1$, (ii) $\mathbf{s}_1 + \mathbf{s}_2$, (iii) $\mathbf{s}_1 + \mathbf{s}_3$, (iv) $\mathbf{s}_1 + \mathbf{s}_5$? Explain.

Solution:

(a) The eigenvalues of A are 0, 1, and 4 with algebraic multiplicities 2, 3, and 1. Eigenvalues 0 and 4 have geometric multiplicity of 1, while eigenvalue 1 has geometric multiplicity of 2.

(b) A is singular since one eigenvalue is 0.

(c) The matrix A is not complete since eigenvalue 1 has algebraic multiplicity 3 and geometric multiplicity 1 and eigenvalue 0 has algebraic multiplicity of 2 and geometric multiplicity 1.

(d) Vectors $\mathbf{s}_1$ and $\mathbf{s}_1 + \mathbf{s}_3$ are eigenvectors while $\mathbf{s}_1 + \mathbf{s}_2$ and $\mathbf{s}_1 + \mathbf{s}_5$ are not. Vector $\mathbf{s}_1$ is the eigenvector that starts the Jordan chain, while $\mathbf{s}_1 + \mathbf{s}_3$ is adding 2 eigenvectors with the same eigenvalue together, and so is itself an eigenvector.

Vector $\mathbf{s}_1 + \mathbf{s}_2$ is a generalized eigenvector with eigenvalue 1, while $\mathbf{s}_1 + \mathbf{s}_5$ adds together two eigenvectors with different eigenvalues, which is not an eigenvvector.

3. (20 points) Let $p(\mathbf{x}) = 5x^2 - 6xy - 2xz + 2y^2 + 2z^2 - 4x + 2y + 2z$

   (a) Show that $p(\mathbf{x})$ has a minimum value.

   (b) Find all the minimizers of $p(\mathbf{x})$

   (c) Find the minimum value of $p(\mathbf{x})$

Solution:

(a) We let $p(\mathbf{x}) = \mathbf{x}^T K\mathbf{x} - 2\mathbf{x}^T\mathbf{f} + c$ and read off the K matrix and $\mathbf{f}$ vector:

$$K = \begin{pmatrix} 5 & -3 & -1 \\ -3 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}$$

$$\mathbf{f} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$$

We need to show that either K is positive definite or that K is positive semi-definite with $\mathbf{f} \in \operatorname{img} K$. We solve $K\mathbf{x} = \mathbf{f}$ using row reduction as this will allow us to check both conditions at once:

$$(K|\mathbf{f}) = \left(\begin{array}{ccc|c} 5 & -3 & -1 & 2 \\ -3 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 5 & -3 & -1 & 2 \\ 0 & 1/5 & -3/5 & 1/5 \\ 0 & -3/5 & 9/5 & -3/5 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 5 & -3 & -1 & 2 \\ 0 & 1/5 & -3/5 & 1/5 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

We see that K is positive semi-definite and that $K\mathbf{x} = \mathbf{f}$ has a solution. Therefore, $p(\mathbf{x})$ has a minimum.

(b) To find the minimizers, we continue solving $K\mathbf{x} = \mathbf{f}$:

$$(K|\mathbf{f}) \to \left(\begin{array}{ccc|c} 5 & -3 & -1 & 2 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 5 & 0 & -10 & 5 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

This system has the particular solution

$$\mathbf{x}^* = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

and homogeneous solution

$$\mathbf{z} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

So the minimizers are all the vectors

$$\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} t$$

for $t \in \mathbb{R}$.

4. (20 points) Let A be the matrix with the SVD

$$A = \begin{pmatrix} 1/\sqrt{6} & -1/\sqrt{3} \\ 2/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{6} & 1/\sqrt{3} \end{pmatrix} \begin{pmatrix} 2\sqrt{30} & 0 \\ 0 & \sqrt{15} \end{pmatrix} \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}$$

(a) What is rank A? Justify your answer.

(b) Find the best rank 1 approximation of A.

(c) Find A^+.

Solution:

(a) As there are two singular values, the rank $A = 2$

(b) Using the singular vectors with the largest singular value gives

$$2\sqrt{30} \begin{pmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{pmatrix} \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 4 & 8 \\ -2 & -4 \end{pmatrix}$$

(c) If the reduced SVD $A = P\Sigma Q^T$ then we have

$$\begin{aligned} A^+ &= Q\Sigma^{-1}P^T \\ &= \begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix} \begin{pmatrix} 1/2\sqrt{30} & 0 \\ 0 & 1/\sqrt{15} \end{pmatrix} \begin{pmatrix} 1/\sqrt{6} & 2/\sqrt{6} & -1/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix} \\ &= \begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix} \begin{pmatrix} 1/12\sqrt{5} & 2/12\sqrt{5} & -1/12\sqrt{5} \\ -1/3\sqrt{5} & 1/3\sqrt{5} & 1/3\sqrt{5} \end{pmatrix} \\ &= \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \frac{1}{12\sqrt{5}} \begin{pmatrix} 1 & 2 & -1 \\ -4 & 4 & 4 \end{pmatrix} \\ A^+ &= \frac{1}{60} \begin{pmatrix} 9 & -6 & -9 \\ -2 & 8 & 2 \end{pmatrix} \end{aligned}$$

5. (20 points) The 2×2 matrix A has eigenvalue $\lambda_1 = 2$ with eigenvector $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and eigenvalue $\lambda_2 = -1$ with eigenvector $\mathbf{v}_2 = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$.

   (a) What is the matrix A?

   (b) Find e^{At}.

Solution:

(a) We calculate A from its diagonalization.

$$\Lambda = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$$

$$S = \begin{pmatrix} 1 & 3 \\ -1 & -2 \end{pmatrix}$$

$$S^{-1} = \begin{pmatrix} -2 & -3 \\ 1 & 1 \end{pmatrix}$$

$$\begin{aligned} A &= \begin{pmatrix} 1 & 3 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -2 & -3 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 3 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} -4 & -6 \\ -1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} -7 & -9 \\ 6 & 8 \end{pmatrix} \end{aligned}$$

(b) Since we already have the diagonalization, we only need to calculate:

$$\begin{aligned} e^{At} &= Se^{\Lambda t}S^{-1} \\ &= \begin{pmatrix} 1 & 3 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{pmatrix}\begin{pmatrix} -2 & -3 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 3 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} -2e^{2t} & -3e^{2t} \\ e^{-t} & e^{-t} \end{pmatrix} \\ &= \begin{pmatrix} -2e^{2t} + 3e^{-t} & -3e^{2t} + 3e^{-t} \\ 2e^{2t} - 2e^{-t} & 3e^{2t} - 2e^{-t} \end{pmatrix} \end{aligned}$$

6. (20 points) Let $\mathbf{x} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$ and S be the set of vectors $\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

   (a) Are the vectors in S linearly independent?

   (b) Find the projection of $\mathbf{x}$ onto the span of S.

Solution:

(a) No, the vectors are not linearly independent. Combining them into a matrix A and reducing it gives

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix} \to \begin{pmatrix} 2 & 1 & 1 \\ 0 & 3/2 & 1/2 \\ 0 & 0 & 0 \end{pmatrix}$$

We see that the last vector in our set was a linear combination of the first two, so S is a set of linearly dependent vectors.

(b) We exclude the last vector from our basis as it is a linear combination of the first two. The remaining vectors aren't orthogonal, so we can't use our projection formula yet until we perform Gram-Schmidt on them:

$$\begin{aligned} \mathbf{w}_1 &= \mathbf{a}_1 \\ \mathbf{w}_2 &= \mathbf{a}_2 - \frac{\mathbf{w}_1^T \mathbf{a}_2}{\mathbf{w}_1^T \mathbf{w}_1} \mathbf{w}_1 = \mathbf{a}_2 - \frac{6}{6} \mathbf{w}_1 \\ &= \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \end{aligned}$$

We now project onto our new orthogonal basis:

$$\begin{aligned} \mathbf{v} &= \frac{\mathbf{x}^T \mathbf{w}_1}{\mathbf{w}_1^T \mathbf{w}_1} \mathbf{w}_1 + \frac{\mathbf{x}^T \mathbf{w}_2}{\mathbf{w}_2^T \mathbf{w}_2} \mathbf{w}_2 = \frac{15}{6} \mathbf{w}_1 + \frac{6}{3} \mathbf{w}_2 \\ &= \frac{5}{2} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 6 \\ 9 \\ 9 \end{pmatrix} \end{aligned}$$

7. (20 points) Let $A = \begin{pmatrix} 1 & 2 \\ -3 & -1 \\ 2 & -1 \end{pmatrix}$

   (a) Find a basis for each of the four fundamental subspaces of A

   (b) Use the Fredholm alternative to determine if $A\mathbf{x} = \mathbf{b}$ has a solution when $\mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$.

Solution:

(a) We find the REF of A:

$$A = \begin{pmatrix} 1 & 2 \\ -3 & -1 \\ 2 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 \\ 0 & 5 \\ 0 & -5 \end{pmatrix} \to \begin{pmatrix} 1 & 2 \\ 0 & 5 \\ 0 & 0 \end{pmatrix}$$

We see that A has full column rank, so the columns are a basis for the image and the kernel is just the zero vector:

$$\operatorname{img} A = \left\{ \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \right\} \qquad \ker A = \{\mathbf{0}\}$$

We can use the transposed non-zero rows of the REF as a basis for the coimage:

$$\operatorname{coimg} A = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 5 \end{pmatrix} \right\}$$

For the cokernel, we need to find the kernel of A^T:

$$A^T = \begin{pmatrix} 1 & -3 & 2 \\ 2 & -1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & -3 & 2 \\ 0 & 5 & -5 \end{pmatrix}$$

Solving the homogeneous equation gives us our basis vector:

$$\operatorname{coker} A = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

(b) We check to see if $\mathbf{b}$ is orthogonal to the cokernel's basis vector:

$$\begin{pmatrix} 2 & 1 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 1 \neq 0$$

So $A\mathbf{x} = \mathbf{b}$ does **not** have a solution.