Write your name and section number below. This exam has 7 problems and is worth 150 points. On each problem, you must show all your work to receive credit on that problem. You are allowed to use one handwritten 8 1/2 x 11 inch piece of paper (front and back) on this exam. You are NOT allowed to use any other notes, books, calculators, or electronic devices.

After you finish the exam, go to the designated area of the room to scan and upload your exam to Gradescope. **Please be sure to match your work with the corresponding problem.** Do not leave the room until you verify that your exam has been correctly uploaded.

Name:

Instructor/Section (Mitchell-002):

1. For each of the following, provide a brief proof or justification.

   (a) (6 points) Let $\mathbf{x} \in \mathbb{C}^n$. Is $\sqrt{\mathbf{x}^T\mathbf{x}}$ a valid norm?

   (b) (6 points) Let $A = \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}$. Does the system $A\mathbf{x} = \mathbf{b}$ have a unique least squares solution?

   (c) (6 points) Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ all be vectors in the same inner product space. If $\langle \mathbf{u}, \mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle$ is it necessarily true that $\mathbf{u} = \mathbf{v}$?

   (d) (6 points) Let A^+ be the pseudo-inverse of A. Does $A^+AA^+ = A^+$?

   (e) (6 points) Does whether a set of vectors is linearly independent depend on the specific choice of inner product?

2. (20 points) Suppose $A = SJS^{-1}$, the Jordan decomposition of matrix A, and matrix J is given by

$$J = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) List all eigenvalues of A, together with their associated algebraic and geometric multiplicities.

(b) Is A singular? Why or why not?

(c) Is A complete? Why or why not?

(d) If $\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_6$ are the columns of S, which of the following, if any, are eigenvectors of A: (i) $\mathbf{s}_1$, (ii) $\mathbf{s}_1 + \mathbf{s}_2$, (iii) $\mathbf{s}_1 + \mathbf{s}_3$, (iv) $\mathbf{s}_1 + \mathbf{s}_5$? Explain.

3. (20 points) Let $p(\mathbf{x}) = 5x^2 - 6xy - 2xz + 2y^2 + 2z^2 - 4x + 2y + 2z$

   (a) Show that $p(\mathbf{x})$ has a minimum value.

   (b) Find all the minimizers of $p(\mathbf{x})$

   (c) Find the minimum value of $p(\mathbf{x})$

4. (20 points) Let A be the matrix with the SVD

$$A = \begin{pmatrix} 1/\sqrt{6} & -1/\sqrt{3} \\ 2/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{6} & 1/\sqrt{3} \end{pmatrix} \begin{pmatrix} 2\sqrt{30} & 0 \\ 0 & \sqrt{15} \end{pmatrix} \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}$$

(a) What is rank A? Justify your answer.

(b) Find the best rank 1 approximation of A.

(c) Find A^+.

5. (20 points) The 2×2 matrix A has eigenvalue $\lambda_1 = 2$ with eigenvector $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and eigenvalue $\lambda_2 = -1$ with eigenvector $\mathbf{v}_2 = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$.

   (a) What is the matrix A?

   (b) Find e^{At}.

6. (20 points) Let $\mathbf{x} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$ and S be the set of vectors $\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

   (a) Are the vectors in S linearly independent?

   (b) Find the projection of $\mathbf{x}$ onto the span of S.

7. (20 points) Let $A = \begin{pmatrix} 1 & 2 \\ -3 & -1 \\ 2 & -1 \end{pmatrix}$

   (a) Find a basis for each of the four fundamental subspaces of A

   (b) Use the Fredholm alternative to determine if $A\mathbf{x} = \mathbf{b}$ has a solution when $\mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$.