1. (**20pts**) Consider $g(x) = \dfrac{-3}{(x+4)^2}$

(a) (6pts) Is $g(x)$ even, odd, or neither? Justify your answer mathematically.

Solution: To check if g is even, odd, or neither, we will calculate $g(-x)$.

$$\begin{aligned} g(-x) &= \frac{-3}{(-x+4)^2} \\ &= \frac{-3}{x^2 - 8x + 16} \end{aligned}$$

We note that, after expanding the denominator,

$$g(x) = \frac{-3}{x^2 + 8x + 16} \text{ and } -g(x) = \frac{3}{x^2 + 8x + 16}$$

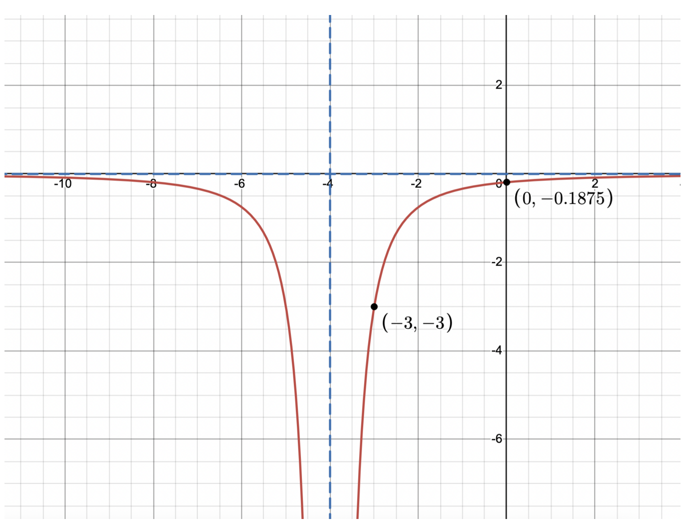
So, since $g(-x) \neq g(x)$ and $g(x) \neq -g(x)$, it is neither odd nor even.

(b) (6pts) State the domain and range of $g(x)$ in interval notation. No explanation is required.

Solution:
Domain: $(-\infty, -4) \cup (-4, \infty)$
Range: $(-\infty, 0)$

(c) (8pts) Sketch a graph of $y = g(x)$. Be sure to **clearly label** the coordinates of all asymptotes and intercepts on your graph.

Solution:
This is a transformation of the base graph $y = \frac{1}{x^2}$. The -3 in the numerator indicates a reflection over the x axis and a vertical stretch by a factor of 3. The $(x+4)^2$ in the denominator indicates a translation, 4 units to the left. So, we know there is a horizontal asymptote at $y = 0$ and a vertical asymptote at $x = -4$, while the y intercept is $\left(0, -\frac{3}{16}\right)$.



2. (**18pts**) Evaluate the following derivatives.

(a) (10pts) Compute $f'(2)$ if $f(x) = 7x^2$. Use the *limit definition of the derivative.*

Solution:

The limit definition of $f'(a)$ is $\lim_{h\to 0} \dfrac{f(a+h) - f(a)}{h}$.

In this problem, $a = 2$ and our $f(x) = 7x^2$, so we set up:

$$\begin{aligned} f'(2) &= \lim_{h\to 0} \frac{7(2+h)^2 - 7(2)^2}{h} \\ &= \lim_{h\to 0} \frac{7(4+4h+h^2) - 28}{h} \\ &= \lim_{h\to 0} \frac{28 + 28h + 7h^2 - 28}{h} \\ &= \lim_{h\to 0} \frac{h(28+7h)}{h} \\ &= \lim_{h\to 0} (28+7h) \\ &= \boxed{28} \end{aligned}$$

Also note that the answer can be verified using the power rule.

(b) (8pts) Find $h'(-1)$ if $h(t) = -(8t + t^3)^2$. Use any differentiation rules.

Solution: We start by finding $h'(t)$ using the chain rule, and then we plug in $t = -1$.

$$\begin{aligned} h'(t) &= -2(8t + t^3) \cdot (8 + 3t^2) \\ h'(-1) &= -2[8(-1) + (-1)^3] \cdot [8 + 3(-1)^2] \\ &= -2 \cdot (-9) \cdot (11) \\ &= \boxed{198} \end{aligned}$$

3. (**20pts**) Consider the function $f(x) = \tan x - \csc x$

(a) (12pts) Evaluate $f'(x)$ when $x = \dfrac{\pi}{3}$

Solution: We first compute $f'(x)$ and then plug in $x = \frac{\pi}{3}$.

$$\begin{aligned} f'(x) &= \sec^2 x - (-\csc x \cot x) \\ &= \sec^2 x + \csc x \cot x \\ f'\left(\frac{\pi}{3}\right) &= (2)^2 + \frac{2}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} \\ &= 4 + \frac{2}{3} \\ &= \boxed{\frac{14}{3}} \end{aligned}$$

(b) (8pts) Find the equation of the line tangent to the graph of $y = f(x)$ at $x = \dfrac{\pi}{3}$. You may leave your answer in point-slope form if you wish.

Solution: In part (a), we found the slope of this tangent line. Now, we need a point on the line. That point is $(\frac{\pi}{3}, f(\frac{\pi}{3}))$.

$$\begin{aligned} f\left(\frac{\pi}{3}\right) &= \tan\left(\frac{\pi}{3}\right) - \csc\left(\frac{\pi}{3}\right) \\ &= \sqrt{3} - \frac{2}{\sqrt{3}} \\ &= \boxed{\frac{1}{\sqrt{3}}} \end{aligned}$$

So, in point slope form, the equation of our tangent line is:

$$\boxed{\left(y - \frac{1}{\sqrt{3}}\right) = \frac{14}{3}\left(x - \frac{\pi}{3}\right)}$$

4. (**18pts**) For the following problem, consider the function

$$f(x) = \begin{cases} \dfrac{x^2+4x-21}{x^2-x-6} & \text{if } x < 3 \\ 2 & \text{if } x = 3 \\ \dfrac{x-3}{\sqrt{3x}-3} & \text{if } x > 3 \end{cases}$$

(a) (6pts) State the mathematical definition of continuity.

Solution: $\lim_{x \to a} f(x) = f(a)$

(b) (12pts) Use your definition from part (a) to determine if the function $f(x)$ is continuous when $x = 3$.

Solution: We want to determine if $\lim_{x \to 3} f(x) = f(3)$. Since our function is defined piecewise, we will take one-sided limits and compare.
From the left:

$$\begin{aligned} \lim_{x \to 3^-} f(x) &= \lim_{x \to 3^-} \frac{x^2+4x-21}{x^2-x-6} \\ &= \lim_{x \to 3^-} \frac{(x-3)(x+7)}{(x-3)(x+2)} \\ &= \lim_{x \to 3^-} \frac{(x+7)}{(x+2)} = \frac{10}{5} \\ &= \boxed{2} \end{aligned}$$

From the right:

$$\begin{aligned} \lim_{x \to 3^+} f(x) &= \lim_{x \to 3^+} \frac{x-3}{\sqrt{3x}-3} \\ &= \lim_{x \to 3^+} \frac{x-3}{\sqrt{3x}-3} \cdot \frac{\sqrt{3x}+3}{\sqrt{3x}+3} \\ &= \lim_{x \to 3^+} \frac{(x-3)(\sqrt{3x}+3)}{3x-9} \\ &= \lim_{x \to 3^+} \frac{(x-3)(\sqrt{3x}+3)}{3(x-3)} \\ &= \lim_{x \to 3^+} \frac{(\sqrt{3x}+3)}{3} = \frac{6}{3} \\ &= \boxed{2} \end{aligned}$$

So, $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = f(3) = 2$, and thus $f(x)$ is continuous at $x = 3$.

5. (**24pts**) For this problem, consider the rational function:

$$R(x) = \frac{-x^2 + 4x + 12}{x^2 + 8x + 12}$$

(a) (16pts) For the following, justify your answers by calculating appropriate limits. You **may not** use dominance of powers arguments.

i. Find all horizontal asymptotes of $R(x)$ if any.

Solution: To find horizontal asymptotes, we take $\lim_{x\to\pm\infty} R(x)$.

$$\begin{aligned}
\lim_{x\to\infty} R(x) &= \lim_{x\to\infty} \frac{-x^2 + 4x + 12}{x^2 + 8x + 12} \\
&= \lim_{x\to\infty} \frac{-x^2 + 4x + 12}{x^2 + 8x + 12} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} \\
&= \lim_{x\to\infty} \frac{-1 + \frac{4}{x} + \frac{12}{x^2}}{1 + \frac{8}{x} + \frac{12}{x^2}} \\
&= -1
\end{aligned}$$

So, we know $R(x)$ has a horizontal asymptote at $y = -1$, but we must also check $\lim_{x\to-\infty} R(x)$.

$$\begin{aligned}
\lim_{x\to-\infty} R(x) &= \lim_{x\to-\infty} \frac{-x^2 + 4x + 12}{x^2 + 8x + 12} \\
&= \lim_{x\to-\infty} \frac{-x^2 + 4x + 12}{x^2 + 8x + 12} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} \\
&= \lim_{x\to-\infty} \frac{-1 + \frac{4}{x} + \frac{12}{x^2}}{1 + \frac{8}{x} + \frac{12}{x^2}} \\
&= -1
\end{aligned}$$

So, $R(x)$ has one $\boxed{\text{horizontal asymptote at } y = -1.}$

ii. Find all vertical asymptotes of $R(x)$ if any.

Solution: Candidates for vertical asymptotes are anywhere the denominator equals 0, so we will factor the denominator.
We note that $x^2 + 8x + 12 = (x + 6)(x + 2)$, and so our candidates for vertical asymptotes are $x = -2$ and $x = -6$. We will check both using limits. Since we only require one of the one-sided limits to approach ∞, we will approach from the left.

$$\begin{aligned}
\lim_{x\to-2^-} R(x) &= \lim_{x\to-2^-} \frac{-x^2 + 4x + 12}{(x + 2)(x + 6)} \\
&= \lim_{x\to-2^-} \frac{-(x + 2)(x - 6)}{(x + 2)(x + 6)}
\end{aligned}$$

$$= \lim_{x \to -2^-} \frac{-(x-6)}{(x+6)}$$

$$= \boxed{-2}$$

Via the same process, we see that $\lim_{x \to -2^+} R(x) = -2$, and since $\lim_{x \to -2}$ exists, there is not a vertical asymptote at $x = -2$.
Now, we will check if there is a vertical asymptote when $x = -6$ by approaching from the left.

$$\lim_{x \to -6^-} R(x) = \lim_{x \to -6^-} \frac{-x^2 + 4x + 12}{(x+2)(x+6)}$$

$$= \lim_{x \to -6^-} \frac{-(x+2)(x-6)}{(x+2)(x+6)}$$

$$= \lim_{x \to -6^-} \frac{-(x-6)}{(x+6)} = \frac{12}{0^-}$$

$$= \boxed{-\infty}$$

This is enough information to conclude that $R(x)$ has one $\boxed{\text{vertical asymptote at } x = -6.}$

iii. Find all removable discontinuities of $R(x)$ if any.

Solution: In part ii, we showed that $\lim_{x \to -2} R(x)$ exists even though $R(-2)$ is undefined. Thus, $R(x)$ has a $\boxed{\text{removable discontinuity at } x = -2.}$

(b) (8pts) Find $R'(0)$.

Solution: We will find $R'(x)$ using the quotient rule and then plug in $x = 0$.

$$R'(x) = \frac{(x^2 + 8x + 12)(-2x + 4) - (-x^2 + 4x + 12)(2x + 8)}{(x^2 + 8x + 12)^2}$$

We **should not** expand this. We will just plug in 0 for each x:

$$R'(0) = \frac{(0^2 + 8 \cdot 0 + 12)(-2 \cdot 0 + 4) - (-0^2 + 4 \cdot 0 + 12)(2 \cdot 0 + 8)}{(0^2 + 8 \cdot 0 + 12)^2}$$

$$= \frac{(12)(4) - (12)(8)}{(12)^2}$$

$$= \frac{4 - 8}{12}$$

$$= \boxed{-\frac{1}{3}}$$