1. (40 pts) Evaluate the following integrals.

(a) $\displaystyle\int \frac{x^2}{\sqrt{1-x^2}}\mathrm{d}x$

(b) $\displaystyle\int_0^{\sqrt{2}} \frac{2x^2-4x+9}{(x+1)(x^2+2)}\mathrm{d}x$

(c) $\displaystyle\int \sin(x)e^{2x}\mathrm{d}x$

(d) $\displaystyle\int_0^{\pi/4} \sec^4(x)\tan^2(x)\mathrm{d}x$

a) $\displaystyle\int \frac{x^2}{\sqrt{1-x^2}}dx$

Let $x = \sin\theta$, $dx = \cos\theta\, d\theta$

$$\int \frac{x^2}{\sqrt{1-x^2}}dx = \int \frac{\sin^2\theta\cos\theta}{\sqrt{1-\sin^2\theta}}d\theta$$

$$= \int \frac{\sin^2\theta\cos\theta\, d\theta}{\sqrt{\cos^2\theta}}$$

$$= \int \sin^2\theta\, d\theta$$

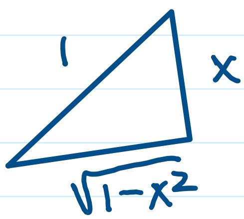
$$= \frac{1}{2}\int 1-\cos 2\theta\, d\theta$$

$$= \frac{1}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right] + C$$

$$= \frac{\theta}{2} - \frac{1}{4}\sin 2\theta + C$$

$$= \frac{\theta}{2} - \frac{1}{4} \cdot 2\sin\theta\cos\theta + C$$

$\theta = \sin^{-1}(x)$ and



(can see this from integral also)

$$\int \frac{x^2}{\sqrt{1-x^2}}\,dx = \frac{1}{2}\sin^{-1}(x) - \frac{1}{2}x\sqrt{1-x^2} + C$$

b) $\displaystyle\int_0^{\sqrt{2}} \frac{2x^2-4x+9}{(x+1)(x^2+2)}\,dx$

PFD:

$$\frac{2x^2-4x+9}{(x+1)(x^2+2)} = \frac{A}{(x+1)} + \frac{Bx+C}{(x^2+2)}$$

$$2x^2-4x+9 = A(x^2+2) + (Bx+C)(x+1)$$

Let $x = -1$:

$$2+4+9 = A(1+2) + (B(-1)+C)(0)$$

$$15 = 3A$$

$$A = 5$$

Then:

$$2x^2-4x+9 = 5x^2+10+Bx^2+Cx+Bx+C$$

$$C+10=9 \implies C=-1$$

$$Bx^2+5x^2=2x^2 \implies B=-3$$

$$-3-1=-4 \checkmark$$

$$A=5, \quad B=-3, \quad C=-1$$

$$\int_0^{\sqrt{2}} \frac{2x^2-4x+9}{(x+1)(x^2+2)}\,dx = \int_0^{\sqrt{2}} \frac{5}{(x+1)} + \frac{-3x-1}{x^2+2}\,dx$$

$$= 5\int_0^{\sqrt{2}} \frac{1}{x+1}\,dx - 3\int_0^{\sqrt{2}} \frac{x}{x^2+2}\,dx - \int_0^{\sqrt{2}} \frac{1}{x^2+2}\,dx$$

middle: let $u=x^2$, $\frac{du}{2}=x\,dx$

bottom bound $=0$, top bound $=(\sqrt{2})^2=2$

$$= 5\ln|x+1|\Big|_0^{\sqrt{2}} - \frac{3}{2}\ln|u+2|\Big|_0^{2}$$
$$- \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)\Big|_0^{\sqrt{2}}$$

$$= 5\ln|\sqrt{2}+1| - \frac{3}{2}\ln|4| + \frac{3}{2}\ln|2|$$
$$- \frac{1}{\sqrt{2}}\cdot\frac{\pi}{4}$$

$$= \ln\left[(\sqrt{2}+1)^5\right] - \ln(4^{3/2}) + \ln(2^{3/2})$$
$$- \frac{\pi}{4\sqrt{2}}$$

$$= \ln\left(\frac{[\sqrt{2}+1]^5\sqrt{8}}{\ldots}\right) - \frac{\pi}{4\sqrt{2}}$$

$$= \ln\left(\frac{[\sqrt{2}+1]^5}{8}\sqrt{8}\right) - \frac{\pi}{4\sqrt{2}}$$

$$= \ln\left(\frac{[\sqrt{2}+1]^5}{\sqrt{8}}\right) - \frac{\pi}{4\sqrt{2}}$$

c) $\int \sin(x) e^{2x}\,dx$

I. B. P., $u = \sin(x)$, $du = \cos(x)\,dx$
$dv = e^{2x}dx$ $v = \frac{1}{2}e^{2x}$

$$\int \sin(x)e^{2x}\,dx = \frac{\sin(x)e^{2x}}{2} - \int \frac{1}{2}e^{2x}\cos(x)\,dx$$

Repeat, $u = \cos(x)$, $du = -\sin(x)\,dx$
$dv = \frac{1}{2}e^{2x}dx$, $v = \frac{1}{4}e^{2x}$

$$\int \sin(x)e^{2x}\,dx = \frac{\sin(x)e^{2x}}{2} - \left[\frac{\cos(x)e^{2x}}{4} + \int \frac{1}{4}e^{2x}\sin(x)\,dx\right]$$

$$= \frac{\sin(x)e^{2x}}{2} - \frac{\cos(x)e^{2x}}{4} - \frac{1}{4}\int \sin(x)e^{2x}\,dx$$

$$\frac{5}{4}\int \sin(x)e^{2x}\,dx = \frac{\sin(x)e^{2x}}{2} - \frac{\cos(x)e^{2x}}{4}$$

$$\int \sin(x)e^{2x}\,dx = \frac{2}{5}e^{2x}\left(\sin(x) - \frac{\cos(x)}{2}\right) + C$$

d) $\int_0^{\pi/4} \sec^4(x)\tan^2(x)\,dx$

$$= \int^{\pi/4} \sec^2(x)\tan^2(x)\sec^2(x)\,dx$$

$$= \int_0^{\pi/4} \sec^2(x)\tan^2(x)\sec^2(x)\,dx$$

$$= \int_0^{\pi/4} (1+\tan^2(x))\tan^2(x)\sec^2(x)\,dx$$

Let $u = \tan(x)$, $du = \sec^2(x)\,dx$

top: $\tan(\pi/4) = 1$ bottom: $\tan(0) = 0$

$$= \int_0^1 (1+u^2)u^2\,du$$

$$= \int_0^1 u^2 + u^4\,du$$

$$= \frac{1}{3}u^3 + \frac{1}{5}u^5 \Big|_0^1$$

$$= \frac{1}{3} + \frac{1}{5} = \frac{8}{15}$$

2. (16 pts) Are the following integrals convergent or divergent? Justify your answers. If convergent, evaluate the integral if possible.

(a) $\displaystyle\int_1^\infty \frac{|\cos(x)|}{x^{3/2}}\mathrm{d}x$

(b) $\displaystyle\int_0^3 \frac{1}{(x-1)^{4/3}}\mathrm{d}x$

a) $\displaystyle\int_1^\infty \frac{|\cos(x)|}{x^{3/2}}dx$

Use comparison thm, unclear how to integrate $|\cos(x)|$.

Since

$$0 \le |\cos(x)| \le 1,$$

$$0 \le \frac{|\cos(x)|}{x^{3/2}} \le \frac{1}{x^{3/2}}$$

$\int_1^\infty \frac{1}{x^{3/2}}\,dx$ is a p-integral w/ $p > 1$

thus convergent

Since $0 \le \frac{|\cos(x)|}{x^{3/2}} \le \frac{1}{x^{3/2}}$,

By comparison, this converges.

b) $$\int_0^3 \frac{1}{(x-1)^{4/3}} = \int_0^1 \frac{1}{(x-1)^{4/3}}\,dx + \int_1^3 \frac{1}{(x-1)^{4/3}}\,dx$$

$$= \lim_{t \to 1^-} \int_0^t \frac{1}{(x-1)^{4/3}}\,dx + \lim_{s \to 1^+} \int_s^3 \frac{1}{(x-1)^{4/3}}\,dx$$

$$= -3(x-1)^{-1/3} \Big|_0^t + -3(x-1)^{-1/3} \Big|_s^3$$

$$\left(\lim_{t \to 1^-} \frac{-3}{(x-1)^{1/3}} - 3\right)$$

goes to infinity: $-3/$sm neg

goes ↗ to infinity: $-3/$ sm neg

Since one part diverges, the integral diverges

3. (36 pts) Let $I = \int_0^\pi \cos^2 \theta d\theta$.

   (a) Estimate I with T_4, the trapezoidal rule with $n = 4$.
   (b) Find a reasonable bound for $|E_T|$ for your calculation in part (a).
   (c) What is the minimum number of intervals required so the error of T_n is less than $\frac{1}{100}$?
   (d) Compute the exact value of I. What is the true error of your estimate in part (a)?
   (e) Is there is a large difference between your bound for $|E_T|$ and the true error? Why might this occur?

a) $\Delta x = \frac{\pi - 0}{4} = \frac{\pi}{4}$

x	0	$\pi/4$	$\pi/2$	$\frac{3\pi}{4}$	π
$f(x)$	1	$\frac{1}{2}$	0	1/2	1
$2f(x)$		1	0	1	

$$T_4 = \frac{\pi}{4} \cdot \frac{1}{2}(1 + 1 + 0 + 1 + 1)$$

$$T_4 = \frac{\pi}{8} \cdot 4 = \frac{\pi}{2}$$

b) $f(x) = \cos^2(x)$

$f'(x) = -2\cos(x)\sin(x)$

$f''(x) = -2\sin^2(x) - 2\cos^2(x)$

$|f''(x)| = |-2\sin^2(x) - 2\cos^2(x)|$

$$\le |2|\,|\sin^2(x)| + |2|\,|\cos^2(x)|$$

$$\le 2\cdot 1 + 2\cdot 1$$

Let $K=4$. Then:

$$|E_T| = \frac{4(\pi)^3}{12\cdot 4^2} = \frac{\pi^3}{48}$$

Alternate bound:

Note $f'(x) = 2\sin(x)\cos(x) = \sin(2x)$

then $f''(x) = 2\cos(2x)$

and $K=2$

(Any reasonable bound is correct)

c)

$$\frac{4\pi^3}{12n^2} < \frac{1}{100} \Rightarrow \frac{400\pi^3}{12} < n^2$$

$$\Rightarrow \sqrt{\frac{100\pi^3}{3}} < n$$

$$\Rightarrow n > 10\pi^2\sqrt{\frac{\pi}{3}}$$

(or anything consistent w/ K in pt b)

d)

$$I = \int_0^{\pi} \cos^2\theta \, d\theta$$

$$= \frac{1}{2}\int_0^{\pi} 1 + \cos 2\theta \, d\theta$$

$$= \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta \Big|_0^{\pi}$$

$$= \frac{\pi}{2} + \frac{1}{4}\sin(2\pi) - \left(\frac{1}{2}(0) + \frac{1}{4}\sin(0)\right)$$

$$= \pi/2$$

The true error is <u>zero</u>.

e) This occurs because $|E_T|$ is an <u>upper bound</u> for the error. It is a worst case. True error may be (much) lower than the estimate.