

1. [2350/061325 (12 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given. Please write your answers in a single column separate from any work you do to arrive at the answer.

- (a) $(\mathbf{u} \cdot \mathbf{v}) \times \mathbf{w} = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$
- (b) The cross product of two nonzero vectors that are scalar multiples of each other has magnitude 0.
- (c) $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$
- (d) The intersection of the plane $z = 2$ and the surface $4x^2 + y^2 + 4z^2 - 4y - 24z + 36 = 0$ is an ellipse.
- (e) The normal component of the acceleration of a particle moving along a straight line is always zero.
- (f) The unit binormal vector $\mathbf{B}$ for a curve lying in the plane $z = 3$ is $\pm \mathbf{k}$.

SOLUTION:

- (a) **FALSE** cross product of a scalar and a vector is not defined
- (b) **TRUE** vectors that are scalar multiples of one another are parallel so the angle θ between them is zero

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\|\sin 0 = 0$$

- (c) **FALSE**

$$(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = \mathbf{u} \cdot \mathbf{u} - \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{u} - \mathbf{v} \cdot \mathbf{v} = \|\mathbf{u}\|^2 - \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{v} - \|\mathbf{v}\|^2 = \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2$$

- (d) **FALSE** Completing the square in the original equation of the surface yields

$$x^2 + \left[\frac{(y-2)}{2} \right]^2 + (z-3)^2 = 1$$

showing that the surface is an ellipsoid centered at $(0, 2, 3)$. If $z = 2$, $x^2 + \left[\frac{(y-2)}{2} \right]^2 = 0$, the only solution of which is $x = 0$, $y = 2$. Thus the trace (intersection) is the single point $(0, 2, 2)$.

- (e) **TRUE** The curvature of a straight line is 0 and since the normal component of the acceleration is proportional to the curvature it, too, is 0.
- (f) **TRUE** $\mathbf{T}$ and $\mathbf{N}$ lie in the plane $z = 3$ and thus have only $\mathbf{i}$ and $\mathbf{j}$ components. Their cross product will then only have a $\mathbf{k}$ component, which is a unit vector, the unit binormal.



2. [2350/061325 (16 pts)] A particle travels along the helix given by $\mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + t \mathbf{k}$. At time $t = \pi$ the particle leaves the path and flies off on a tangent. Find the location of the particle at $t = 2\pi$ assuming no forces act on it after it leaves the helix.

SOLUTION:

The tangent vector to the helix is $\mathbf{r}'(t) = -\sin t \mathbf{i} + \cos t \mathbf{j} + \mathbf{k}$. The particle leaves the helix at $\mathbf{r}(\pi) = -\mathbf{i} + \pi \mathbf{k}$ and moves in the direction of the tangent vector at this point, namely $\mathbf{r}'(\pi) = -\mathbf{j} + \mathbf{k}$. Since no other forces are acting on the particle, it moves in this direction for $t \geq \pi$. This trajectory is a straight line through the point $(-1, 0, \pi)$ in the direction of $-\mathbf{j} + \mathbf{k}$. The equation of this line is $\ell(t) = \langle -1, 0, \pi \rangle + (t - \pi) \langle 0, -1, 1 \rangle$ for $t \geq \pi$. When $t = 2\pi$, the particle is at $\ell(2\pi) = \langle -1, 0, \pi \rangle + (2\pi - \pi) \langle 0, -1, 1 \rangle = \langle -1, -\pi, 2\pi \rangle$.



3. [2350/061325 (10 pts)] Find the equation of, and identify, the quadric surface whose points are equidistant from the point $P_0(2, 0, 0)$ and the plane containing the point $(-2, 0, 0)$ whose normal vector is $\mathbf{i}$.

SOLUTION:

The equation of the plane is $x = -2$. Let $P(x, y, z)$ be a point on the surface. The distant from P to the plane is $\sqrt{(x+2)^2}$ and the distance from P to P_0 is $\sqrt{(x-2)^2 + y^2 + z^2}$. Equating these two distances gives

$$\begin{aligned}\sqrt{(x+2)^2} &= \sqrt{(x-2)^2 + y^2 + z^2} \\ x^2 + 4x + 4 &= x^2 - 4x + 4 + y^2 + z^2 \\ x &= \frac{1}{8}(y^2 + z^2)\end{aligned}$$

The surface is a (circular) paraboloid. ■

4. [2350/061325 (20 pts)] Consider the vector function $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ with $-\infty < t < \infty$.

- (a) (10 pts) Compute the *torsion*, τ (the measure of the degree of twisting of a curve), given by $\tau = \frac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{\|\mathbf{r}' \times \mathbf{r}''\|^2}$ at the point $(2, 4, 8)$.
- (b) (10 pts) Are there any points on the curve where the velocity and acceleration vectors are orthogonal? If so, find them. If not, explain why not.

SOLUTION:

- (a) $\mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle$, $\mathbf{r}''(t) = \langle 0, 2, 6t \rangle$ and $\mathbf{r}'''(t) = \langle 0, 0, 6 \rangle$. The curve passes through the point $(2, 4, 8)$ when $t = 2$ so that

$$\mathbf{r}'(2) \times \mathbf{r}''(2) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & 12 \\ 0 & 2 & 12 \end{vmatrix} = \langle 24, -12, 2 \rangle \implies \|\mathbf{r}'(2) \times \mathbf{r}''(2)\| = \sqrt{24^2 + (-12)^2 + 2^2} = \sqrt{724}$$

$$[\mathbf{r}'(2) \times \mathbf{r}''(2)] \cdot \mathbf{r}'''(2) = \langle 24, -12, 2 \rangle \cdot \langle 0, 0, 6 \rangle = 12 \implies \tau = \frac{12}{724} = \frac{3}{181}$$

- (b) We need to determine if there are any values of t where $\mathbf{r}'(t) \cdot \mathbf{r}''(t) = 0$.

$$\mathbf{r}'(t) \cdot \mathbf{r}''(t) = \langle 1, 2t, 3t^2 \rangle \cdot \langle 0, 2, 6t \rangle = 4t + 18t^3 = 2t(2 + 9t^2) = 0 \implies t = 0$$

The velocity and acceleration vectors are perpendicular when $t = 0$ which is the point $(0, 0, 0)$ on the curve. ■

5. [2350/061325 (10 pts)] Find the position vector $\mathbf{r}(t)$ of an object subject to the following conditions: it undergoes an acceleration of $e^t\mathbf{i} + 2t\mathbf{j} + (t+1)\mathbf{k}$ for $t \geq 0$ and it begins its motion at $\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ with a velocity of $\mathbf{i} + \mathbf{k}$.

SOLUTION:

Integrate the acceleration to find that

$$\mathbf{v}(t) = \int \mathbf{a}(t) dt = \int \mathbf{r}''(t) dt = \mathbf{r}'(t) = \int \langle e^t, 2t, t+1 \rangle dt = \left\langle e^t, t^2, \frac{1}{2}t^2 + t \right\rangle + \mathbf{C}$$

Then $\mathbf{r}'(0) = \langle 1, 0, 1 \rangle = \langle 1, 0, 0 \rangle + \mathbf{C} \implies \mathbf{C} = \langle 0, 0, 1 \rangle \implies \mathbf{r}'(t) = \langle e^t, t^2, \frac{1}{2}t^2 + t + 1 \rangle$. Integrate the velocity to find that

$$\mathbf{r}(t) = \int \mathbf{v}(t) dt = \int \mathbf{r}'(t) dt = \int \left\langle e^t, t^2, \frac{1}{2}t^2 + t + 1 \right\rangle dt = \left\langle e^t, \frac{1}{3}t^3, \frac{1}{6}t^3 + \frac{1}{2}t^2 + t \right\rangle + \mathbf{C}$$

Then $\mathbf{r}(0) = \langle 1, 2, 2 \rangle = \langle 1, 0, 0 \rangle + \mathbf{C} \implies \mathbf{C} = \langle 0, 2, 2 \rangle \implies \mathbf{r}(t) = \langle e^t, \frac{1}{3}t^3 + 2, \frac{1}{6}t^3 + \frac{1}{2}t^2 + t + 2 \rangle$. ■

6. [2350/061325 (24 pts)] Consider the intersecting lines $L_1(t) = \langle 7 - 2t, t, -4 - t \rangle$ and $L_2(s) = \langle 3 + 2s, -3 + 4s, -8 + 3s \rangle$.

- (a) (6 pts) Find the coordinates of the point where the lines intersect.
- (b) (6 pts) Find the equation of the plane containing the lines. Write your final answer in the form $ax + by + cz = d$.
- (c) (6 pts) Find the symmetric equations of the line normal to the plane you found in part (b) and passing through the point you found in part (a).

- (d) (6 pts) Find the coordinates of the point where the line from part (c) intersects the plane $x + y + z = 2$.

SOLUTION:

- (a) Since the lines intersect, we know that there exist s, t such that

$$7 - 2t = 3 + 2s \quad (1)$$

$$t = -3 + 4s \quad (2)$$

$$-4 - t = -8 + 3s \quad (3)$$

Substituting (2) into (1) yields $7 - 2(-3 + 4s) = 3 + 2s \implies s = 1$. Eq. (2) then gives $t = 1$ and (3) is satisfied using these values. The lines intersect at $(5, 1, -5)$.

- (b) A point in the plane is $(5, 1, -5)$. We need a normal vector, which is obtained as the cross product of the direction vectors of the two lines,

$$\langle -2, 1, -1 \rangle \times \langle 2, 4, 3 \rangle = \langle 7, 4, -10 \rangle$$

$$\text{Then } 7(x - 5) + 4(y - 1) - 10(z + 5) = 0 \implies 7x + 4y - 10z = 89$$

- (c) The vector equation of the line is $\mathbf{r}(t) = \langle 5, 1, -5 \rangle + t\langle 7, 4, -10 \rangle$ giving the symmetric equations

$$\frac{x - 5}{7} = \frac{y - 1}{4} = \frac{z + 5}{-10}$$

- (d) The point where the line intersects the plane is found by substituting the parametric equations of the line into the plane's equation to find the value of t at the intersection point:

$$5 + 7t + 1 + 4t - 5 - 10t = 2 \implies t = 1$$

giving the point of intersection as $(12, 5, -15)$.



7. [2350/061325 (8 pts)] Consider the vector function $\mathbf{r}(t) = \langle t^2, \sin t - t \cos t, \cos t + t \sin t \rangle$ for $0 \leq t \leq c$. Find the value of c such that the arc length at $t = c$ is $8\sqrt{5}$.

SOLUTION:

Need to find c such that $s(c) = \int_0^c \|\mathbf{r}'(u)\| \, du = 8\sqrt{5}$.

$$\|\mathbf{r}'(t)\| = \|\langle 2t, t \sin t, t \cos t \rangle\| = \sqrt{4t^2 + t^2 \sin^2 t + t^2 \cos^2 t} = \sqrt{5t^2} = \sqrt{5}|t| = \sqrt{5}t \text{ since } t \geq 0$$

Therefore

$$\int_0^c \sqrt{5}t \, dt = 8\sqrt{5} \implies \left. \frac{\sqrt{5}t^2}{2} \right|_0^c = \frac{\sqrt{5}c^2}{2} = 8\sqrt{5} \implies c = 4 \quad (\text{need } c > 0)$$

