1. [2360/061325 (11 pts)] Consider the autonomous differential equation $y' = -(y-10)^2(y-4)$.

   (a) (2 pts) Find all equilibrium solutions of the equation.

   (b) (2 pts) Determine the y values where the solution increases and decreases.

   (c) (2 pts) Determine the stability of the equilibrium solutions.

   (d) (5 pts) Plot the phase line for the differential equation.

   SOLUTION:

   (a) $-(y-10)^2(y-4) = 0 \implies y = 4$ and $y = 10$ are equilibrium solutions.

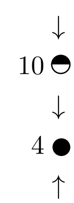
   (b)

$$\begin{aligned} y > 10: &\quad y' < 0 \\ 4 < y < 10: &\quad y' < 0 \\ y < 4: &\quad y' > 0 \end{aligned}$$

   The solution increases for $y < 4$ and decreases for $y \in (4, 10) \cup (10, \infty)$

   (c) $y = 4$ is stable and $y = 10$ is semi-stable.

   (d) Phase line



2. [2360/061325 (11 pts)] Consider the differential equation $\dfrac{dx}{dt} = \dfrac{x}{10} + 5$.

   (a) (8 pts) Find the general solution of the differential equation using the Euler-Lagrange Two Stage (variation of parameters) method. Minimal credit, if any, will be awarded for simply using a formula that yields the result. Instead, show all the steps needed to arrive at the solution.

   (b) (3 pts) Find the solution to differential equation that passes through the point $(0, 0)$.

   SOLUTION:

   (a) Begin by solving the associated homogeneous problem, $\dfrac{dx}{dt} - \dfrac{x}{10} = 0$ with separation of variables.

$$\frac{dx}{dt} = \frac{x}{10}$$

$$\int \frac{dx}{x} = \int \frac{dt}{10}$$

$$\ln|x| = \frac{t}{10} + k$$

$$|x| = e^k e^{t/10}$$

$$x = Ce^{t/10}$$

Replace C with a varying parameter $v(t)$, form the particular solution $x_p = v(t)e^{t/10}$ and substitute this into the DE:

$$x_p' - \frac{x_p}{10} = \frac{v}{10}e^{t/10} + v'e^{t/10} - \frac{ve^{t/10}}{10} = 5$$

$$v(t) = \int 5e^{-t/10}\,\mathrm{d}t = -50e^{-t/10}$$

giving $x_p = -50e^{-t/10}e^{t/10} = -50$. Applying the Nonhomogeneous Principle we have

$$x = x_h + x_p = Ce^{t/10} - 50$$

(b) To pass through the origin requires that $x(0) = 0$. Applying this yields $C - 50 = 0 \implies C = 50$ so that

$$x(t) = 50e^{t/10} - 50$$

■

3. [2360/061325 (36 pts)] Consider the differential equation $y' - \frac{y}{t^2} - 1 = 0$.

(a) (8 pts) On your paper, create a table with the numbers i, ii, iii, iv in it. Next to each number, write the word TRUE or FALSE as appropriate. No work need be shown and no partial credit given on this part.

i. The equation is second order.

ii. The equation is constant coefficient.

iii. The equation is linear.

iv. The equation is homogeneous.

(b) (12 pts) Draw the isoclines corresponding to slopes $-1, 0$, and 1. Be sure to put the appropriate tick marks/line segments on each isocline.

(c) (16 pts) Now consider the initial value problem consisting of the differential equation and the initial condition $y(1) = 0$.

i. (8 pts) What conclusions, if any, can be drawn from Picard's theorem regarding the existence and/or uniqueness of solutions to the initial value problem? Justify your answer.

ii. (8 pts) Use Euler's method with a step size of $h = 0.5$ to estimate the value of $y(2)$.

SOLUTION:

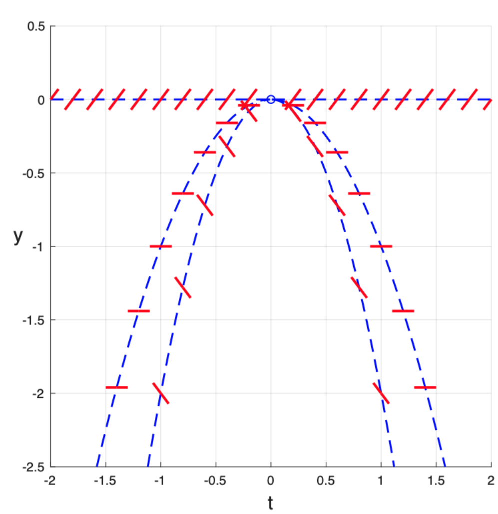
(a) i. FALSE

ii. FALSE

iii. TRUE

iv. FALSE

(b) Note that the isoclines are not defined at $t = 0$ but elsewhere are given by $y = t^2(k - 1)$ where $k = -1, 0, 1$.



(c) i. Rewriting the differential equation as $y' = \frac{y}{t^2} + 1$ yields $f(t,y) = \frac{y}{t^2} + 1$ and $f_y(t,y) = \frac{1}{t^2}$, both of which are continuous in a rectangle around $(1,0)$. Therefore, Picard's Theorem guarantees a unique solution to the initial value problem on an open interval surrounding $t = 1$.

ii.

$$y(1.5) \approx y_1 = y_0 + hf(t_0, y_0) = 0 + 0.5\left(\frac{0}{1^2} + 1\right) = \frac{1}{2}$$

$$y(2.0) \approx y_2 = y_1 + hf(t_1, y_1) = \frac{1}{2} + 0.5\left(\frac{1/2}{(3/2)^2} + 1\right) = \frac{10}{9}$$

■

4. [2360/061325 (7 pts)] Suppose at midnight ($t = 0$ hours) the temperature, T, in your apartment is 72 degrees and the outside temperature is 32 degrees. The outside temperature falls to 16 degrees at 6 AM. The insulation in your apartment is such that the constant of proportionality in Newton's Law of Cooling is $k = \frac{1}{2}$. Assuming that you have no way to heat the apartment, the differential equation describing this situation is

$$\frac{dT}{dt} = 16 - t - \frac{1}{2}T \qquad (1)$$

(a) (2 pts) Show that the general solution of Eq. (1) is $T(t) = 36 - 2t + Ce^{-t/2}$ where C is an arbitrary constant.

(b) (5 pts) What is the temperature in your apartment at 6 AM?

SOLUTION:

(a) Substituting the given solution into the left hand side of (1) results in

$$\frac{dT}{dt} = -2 - \frac{C}{2}e^{-t/2}$$

and substituting into the right hand side yields

$$16 - t - \frac{1}{2}\left(36 - 2t + Ce^{-t/2}\right) = 16 - t - 18 + t - \frac{C}{2}e^{-t/2} = -2 - \frac{C}{2}e^{-t/2}$$

Since these two expressions are the same, the solution as given is the solution to the differential equation. Note that this can also be shown by solving the equation outright.

(b) Using the initial condition of $T = 72$ when $t = 0$ gives $72 = 36 - 2(0) + C(1) \implies C = 36$. Thus

$$T(t) = 36 - 2t + 36e^{-t/2} \implies T(6) = 36 - 12 + 36e^{-3} = 24 + 36e^{-3}$$

■

5. [2360/061325 (9 pts)] Use the Integrating Factor method to find the general solution of the differential equation $t^2y' = -ty + 2$, $t > 0$. Don't simply plug into a formula; show all the steps.

SOLUTION:

Start by getting the differential equation into the correct form as $y' + \frac{1}{t}y = \frac{2}{t^2}$. Then $p(t) = \frac{1}{t}$ and

$$\int p(t)\,dt = \int \frac{dt}{t} = \ln t$$

where the absolute value is not needed since $t > 0$. Then the integrating factor is $\mu(t) = t$ and we have

$$\int (ty)'dt = \int \frac{2}{t}\,dt$$

$$ty = 2\ln t + C \qquad \text{(absolute value not needed since } t > 0\text{)}$$

$$y = \frac{\ln t^2 + C}{t}$$

■

6. [2360/061325 (9 pts)] A tank initially contains 900 liters (L) of pure water. Salt water with a concentration of 2 g/L is pumped into the tank at 3 L/min and the well-mixed solution is drained from the tank at a rate of 1 L/min. Set up, but **do not solve**, the initial value problem (IVP) describing this situation. Be sure to describe your variables.

SOLUTION:

Since the flow rate in differs from the flow rate out, the volume of fluid in the tank will vary with time. Letting $V(t)$ be the volume of liquid in the tank we have

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \text{flow in} - \text{flow out} = 3 - 1 = 2, V(0) = 900 \implies V(t) = 900 + 2t$$

Let $x(t)$ be the amount of salt (g) in the tank at time t. Then the rate of change of salt in the tank being equal to the rate of incoming salt minus the rate of outgoing salt gives

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \text{rate in} - \text{rate out} = 2(3) - \frac{x}{900 + 2t}(1)$$

Since the tank contains pure water initially, $x(0) = 0$. The initial value problem is thus

$$\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{x}{900 + 2t} = 6, \; x(0) = 0$$

■

7. [2360/061325 (6 pts)] Determine the h and v nullclines, and equilibrium points, of the following system of differential equations.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x + y - 1$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = x^2 + y^2 + 2$$

SOLUTION:

The h nullclines occur where $\mathrm{d}y/\mathrm{d}t = x^2 + y^2 + 2 = 0$ which has no real solutions so the system has no h nullclines. The v nullclines occur where $\mathrm{d}x/\mathrm{d}t = x + y - 1 = 0$ which is the line $y = 1 - x$. There are no equilibrium points since $\mathrm{d}x/\mathrm{d}t$ and $\mathrm{d}y/\mathrm{d}t$ are never simultaneously zero. ■

8. [2360/061325 (11 pts)] In the homework, we looked at substitutions that transformed differential equations from something that could not be solved to something that could be solved (for example non-separable to separable; nonlinear to linear). Here we make a substitution that transforms a second order equation into a first order equation. Consider the differential equation $\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = 7 + \frac{\mathrm{d}y}{\mathrm{d}t}$.

   (a) (3 pts) Convert this into a first order equation by using the substitution $v = \mathrm{d}y/\mathrm{d}t$.

   (b) (4 pts) Solve the differential equation in part (a).

   (c) (4 pts) Find the solution, y, of the original differential equation.

SOLUTION:

(a) If $v = \mathrm{d}y/\mathrm{d}t$, $\mathrm{d}v/\mathrm{d}t = \mathrm{d}^2y/\mathrm{d}t^2$ so that the differential equation becomes

$$\frac{\mathrm{d}v}{\mathrm{d}t} = 7 + v$$

(b)

$$\int \frac{\mathrm{d}v}{v + 7}\,\mathrm{d}t = \int \mathrm{d}t$$

$$\ln|v + 7| = t + k \implies |v + 7| = e^k e^t \implies v + 7 = Ce^t \implies v = Ce^t - 7$$

(c)

$$y = \int \frac{\mathrm{d}y}{\mathrm{d}t}\,\mathrm{d}t = \int v\,\mathrm{d}t = \int \left(Ce^t - 7\right)\mathrm{d}t \implies y(t) = Ce^t - 7t + D$$

■