- This exam is worth 100 points and has 8 problems.
- Show all work and simplify your answers! Answers with no justification will receive no points unless otherwise noted.
- Begin each problem on a new page.
- **DO NOT LEAVE THE EXAM UNTIL YOUR HAVE SATISFACTORILY SCANNED AND UPLOADED YOUR EXAM TO GRADESCOPE.**
- You are taking this exam in a proctored and honor code enforced environment. No calculators, cell phones, or other electronic devices or the internet are permitted during the exam. You are allowed one 8.5"$\times$ 11" crib sheet with writing on one side.
- Remote students are allowed use of a computer during the exam only for a live video of their hands and face and to view the exam in the Zoom meeting. Remote students cannot interact with anyone except the proctor during the exam.

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0. At the top of the first page that you will be scanning and uploading to Gradescope, write the following statement and sign your name to it: "I will abide by the CU Boulder Honor Code on this exam." **FAILURE TO INCLUDE THIS STATEMENT AND YOUR SIGNATURE MAY RESULT IN A PENALTY.**

1. [2360/061325 (11 pts)] Consider the autonomous differential equation $y' = -(y-10)^2(y-4)$.

   (a) (2 pts) Find all equilibrium solutions of the equation.

   (b) (2 pts) Determine the y values where the solution increases and decreases.

   (c) (2 pts) Determine the stability of the equilibrium solutions.

   (d) (5 pts) Plot the phase line for the differential equation.

2. [2360/061325 (11 pts)] Consider the differential equation $\dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{x}{10} + 5$.

   (a) (8 pts) Find the general solution of the differential equation using the Euler-Lagrange Two Stage (variation of parameters) method. Minimal credit, if any, will be awarded for simply using a formula that yields the result. Instead, show all the steps needed to arrive at the solution.

   (b) (3 pts) Find the solution to differential equation that passes through the point $(0,0)$.

3. [2360/061325 (36 pts)] Consider the differential equation $y' - \dfrac{y}{t^2} - 1 = 0$.

   (a) (8 pts) On your paper, create a table with the numbers i, ii, iii, iv in it. Next to each number, write the word TRUE or FALSE as appropriate. No work need be shown and no partial credit given on this part.

      i. The equation is second order.

      ii. The equation is constant coefficient.

      iii. The equation is linear.

      iv. The equation is homogeneous.

   (b) (12 pts) Draw the isoclines corresponding to slopes $-1, 0$, and 1. Be sure to put the appropriate tick marks/line segments on each isocline.

   (c) (16 pts) Now consider the initial value problem consisting of the differential equation and the initial condition $y(1) = 0$.

      i. (8 pts) What conclusions, if any, can be drawn from Picard's theorem regarding the existence and/or uniqueness of solutions to the initial value problem? Justify your answer.

      ii. (8 pts) Use Euler's method with a step size of $h = 0.5$ to estimate the value of $y(2)$.

MORE PROBLEMS BELOW/ON REVERSE

4. [2360/061325 (7 pts)] Suppose at midnight ($t = 0$ hours) the temperature, T, in your apartment is 72 degrees and the outside temperature is 32 degrees. The outside temperature falls to 16 degrees at 6 AM. The insulation in your apartment is such that the constant of proportionality in Newton's Law of Cooling is $k = \frac{1}{2}$. Assuming that you have no way to heat the apartment, the differential equation describing this situation is

$$\frac{\mathrm{d}T}{\mathrm{d}t} = 16 - t - \frac{1}{2}T \tag{1}$$

   (a) (2 pts) Show that the general solution of Eq. (1) is $T(t) = 36 - 2t + Ce^{-t/2}$ where C is an arbitrary constant.

   (b) (5 pts) What is the temperature in your apartment at 6 AM?

5. [2360/061325 (9 pts)] Use the Integrating Factor method to find the general solution of the differential equation $t^2y' = -ty + 2$, $t > 0$. Don't simply plug into a formula; show all the steps.

6. [2360/061325 (9 pts)] A tank initially contains 900 liters (L) of pure water. Salt water with a concentration of 2 g/L is pumped into the tank at 3 L/min and the well-mixed solution is drained from the tank at a rate of 1 L/min. Set up, but **do not solve**, the initial value problem (IVP) describing this situation. Be sure to describe your variables.

7. [2360/061325 (6 pts)] Determine the h and v nullclines, and equilibrium points, of the following system of differential equations.

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= x + y - 1\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= x^2 + y^2 + 2\end{aligned}$$

8. [2360/061325 (11 pts)] In the homework, we looked at substitutions that transformed differential equations from something that could not be solved to something that could be solved (for example non-separable to separable; nonlinear to linear). Here we make a substitution that transforms a second order equation into a first order equation. Consider the differential equation $\dfrac{\mathrm{d}^2y}{\mathrm{d}t^2} = 7 + \dfrac{\mathrm{d}y}{\mathrm{d}t}$.

   (a) (3 pts) Convert this into a first order equation by using the substitution $v = \mathrm{d}y/\mathrm{d}t$.

   (b) (4 pts) Solve the differential equation in part (a).

   (c) (4 pts) Find the solution, y, of the original differential equation.