APPM 3570/STAT 3100 Exam 3 Section 002 Spring 2025

NAME: ______________________________

SECTION: (Circle One) 001 at 10:10 am [or] 002 at 2:30 pm

Instructions:

1. Calculators are permitted.
2. Notes, your text and other books, cell phones, and other electronic devices are not permitted—except for calculators or as needed to view and upload your work.
3. Justify your answers, show all work.
4. When you have completed the exam, go to the uploading area in the room and scan your exam and upload it to Gradescope.
5. Don't forget to scan any pages you used for extra space!
6. Verify that everything has been uploaded correctly and the pages have been associated to the correct problems.
7. Turn in your hardcopy exam.

On my honor as a University of Colorado Boulder student, I have neither given nor received unauthorized assistance on this work.   Signature: ______________________ Date: ____________

Duration: 90 minutes

Problem 1. (28 points.) Consider a model in which the joint pdf of city populations x (in millions) obeys a distribution with parameter y, which itself is randomly drawn. The joint density function for X and Y is:

$$f_{X,Y}(x,y) = \begin{cases} y(y+1)\frac{e^{-y}}{x^{y+2}} & 1 \le x < \infty,\ 0 < y < \infty \\ 0 & \text{otherwise} \end{cases}$$

(a) What is the marginal density of Y?

(b) What is the conditional density of X given that $Y = y$?

(c) Determine the conditional expectation $\mathrm{E}[X|Y]$.

(d) Determine the total expectation $\mathrm{E}[X]$.

(e) Are X and Y independent? Justify your answer.

Solution:

(a) (6 points.)

$$\begin{aligned} f_Y(y) &= y(y+1)e^{-y}\int_1^\infty \frac{1}{x^{y+2}}\,dx \\ &= y(y+1)e^{-y}\left(\frac{-1}{(y+1)x^{y+1}}\bigg|_1^\infty\right) \\ &= y(y+1)e^{-y}\left(\frac{1}{y+1}\right) \\ &= ye^{-y} \end{aligned}$$

$$f_Y(y) = \begin{cases} ye^{-y} & y > 0 \\ 0 & \text{otherwise} \end{cases}$$

(b) (6 points.)

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f_{X,Y}(x,y)}{f_Y(y)} \\ &= \frac{y(y+1)\frac{e^{-y}}{x^{y+2}}}{ye^{-y}} \\ &= \frac{y+1}{x^{y+2}} \end{aligned}$$

$$f_{X|Y}(x|y) = \begin{cases} \frac{y+1}{x^{y+2}} & x \ge 1 \\ 0 & \text{otherwise} \end{cases}$$

Here y is a fixed constant greater than zero.

(c) (6 points.)

$$\begin{aligned} \mathrm{E}[X|Y] &= \int_1^\infty x\frac{y+1}{x^{y+2}}\,dx \\ &= \int_1^\infty \frac{y+1}{x^{y+1}}\,dx \\ &= (y+1)\left(-\frac{1}{yx^y}\bigg|_1^\infty\right) \end{aligned}$$

$$= (y+1)\left(\frac{1}{y}\right)$$

$$= \frac{y+1}{y}$$

Here y is a fixed constant greater than zero.

(d) (6 points.)

$$\begin{aligned}
E_X[X] &= E_Y[E_X[X|Y]] \\
&= \int_0^\infty \frac{y+1}{y} y e^{-y}\, dy \\
&= \int_0^\infty (y+1)e^{-y}\, dy \\
&= \int_0^\infty (ye^{-y} + e^{-y})\, dy \\
&= 1+1 \qquad \text{recognizing the mean of an exponential} \\
&\qquad\qquad \text{and the pdf of an exponential} \\
&= 2
\end{aligned}$$

or, equivalently

$$\begin{aligned}
&= \left(-ye^{-y} - e^{-y} - e^{-y}\right)\Big|_0^\infty \\
&= 0+1+1 \\
&= 2
\end{aligned}$$

(e) (4 points.)No, we can see that the joint pdf of X and Y cannot be factored into a function of x and a function of y.

(Use the back if additional space is needed!)

Problem 2. (15 points.) An organism has a lifetime X (in days) distributed exponentially $X \sim \textit{Exponential}(1)$, so $f_X(x) = e^{-x}$ for $x > 0$.

The organism can have a single offspring at time Y (in days) also distributed exponentially $Y \sim \textit{Exponential}(2)$, so $f_Y(y) = 2e^{-2y}$ for $y > 0$.

Notice that if $X < Y$, the organism will have no offspring, and its lineage terminates.

Assume that X and Y are independent.

Determine the probability an organism dies before it has an offspring, $\mathrm{P}(X < Y)$.

Solution:

$$\begin{aligned}
\mathrm{P}(X < Y) &= \int_0^\infty \int_0^y 2e^{-x}e^{-2y}\,dx\,dy \\
&= \int_0^\infty 2e^{-2y}\left(-e^{-x}\Big|_0^y\right)dy \\
&= \int_0^\infty 2e^{-2y} - 2e^{-3y}\,dy \\
&= \left(-e^{-y} + \frac{2}{3}e^{-3y}\right)\Big|_0^\infty \\
&= (0+0) - \left(-1 + \frac{2}{3}\right) \\
&= \frac{1}{3}
\end{aligned}$$

Equivalently,

$$\begin{aligned}
\mathrm{P}(X < Y) &= \int_0^\infty \int_x^\infty 2e^{-x}e^{-2y}\,dy\,dx \\
&= \int_0^\infty 2e^{-x}\left(-\frac{e^{-2y}}{2}\Big|_x^\infty\right)dx \\
&= \int_0^\infty e^{-3x}\,dx \\
&= \left(-\frac{e^{-3x}}{3}\right)\Big|_0^\infty \\
&= \frac{1}{3}
\end{aligned}$$

(Use the back if additional space is needed!)

Problem 3. (22 points.) Suppose a barn has 10 stalls, all in a row. Suppose 6 horses and 4 goats are each led into a stall, randomly and independently, and each stall holds exactly one animal.

What is the expected number of horses that are in a stall next to a goat?

Solution: Let $H_i = 1$ if there is a horse in stall i next to a goat and zero otherwise. Our goal is to find $\text{E}[\sum_{i=1}^{10} H_i] = \sum_{i=1}^{10} \text{E}[H_i]$.

The two end stalls will have the same expectation of holding a horse that is next to a goat.

$\text{E}[H_1] = \text{P}(H_1 = 1) = \text{P}(\text{“horse in stall one and goat in stall two”}) = \frac{6}{10} \cdot \frac{4}{9} = \frac{4}{15}$.
$\text{E}[H_{10}] = \text{P}(H_{10} = 1) = \text{P}(\text{“horse in stall ten and goat in stall nine”}) = \frac{6}{10} \cdot \frac{4}{9} = \frac{4}{15}$.

The other eight stalls will have the same expectation of holding a horse that is next to a goat.

$\text{E}[H_2] = \text{P}(H_2 = 1) = \text{P}(\text{“horse in stall two and goat in stall one and goat in stall three”}$
$\quad \cup$ “horse in stall two and goat in stall one and horse in stall three”
$\quad \cup$ “horse in stall two and horse in stall two and goat in stall three”)

$= \text{P}(\text{“horse in stall two and goat in stall one and goat in stall three”})$
$\quad +\text{P}(\text{“horse in stall two and goat in stall one and horse in stall three”})$
$\quad +\text{P}(\text{“horse in stall two and horse in stall two and goat in stall three”})$(since disjoint events, by axiom 3)
$= \frac{4}{9} \cdot \frac{6}{10} \cdot \frac{3}{8} + \frac{4}{9} \cdot \frac{6}{10} \cdot \frac{5}{8} + \frac{5}{9} \cdot \frac{6}{10} \cdot \frac{4}{8} = \frac{13}{30}$.

A similar calculation can be made for all of the interior stalls, 2 through 9. Therefore, $\text{E}[H_i] = \text{P}(H_i = 1) = \frac{13}{30}$. for $i = 2, 3, 4, 5, 6, 7, 8, 9$.

$\sum_{i=1}^{10} \text{E}[H_i] = 2\left(\frac{4}{15}\right) + 8\left(\frac{13}{30}\right) = 4$
The expected number of horses in a stall next to a goat is four.

(Use the back if additional space is needed!)

Problem 4. (25 points.) A bank teller serves customers standing in a queue one by one. Suppose that the service time in minutes for a customer follows an exponential distribution with $\lambda = \frac{1}{2}$ as the rate parameter. Assume that service times for different bank customers are independent.

(a) Describe the approximate distribution of the total time that the bank teller will spend serving 50 randomly selected customers. Justify any assumptions.

(b) Estimate the probability that the total time that the bank teller will spend serving 50 randomly selected customers is between 90 and 95 minutes.

Solution:

(a) (12 points.) Let X_i be the service time for customer i, for $i = 1, 2, \ldots, 50$. Since each is *Exponential*$(\frac{1}{2})$, then each has mean 2 and variance 4. Let Y denote the total time that the bank teller spends serving 50 randomly selected customers, so $Y = \sum_{i=1}^{50} X_i$. Since $n = 50$ is large enough, by the Central Limit Theorem $Y \overset{approx}{\sim} \mathcal{N}(50 \cdot 2, 50 \cdot 4)$, or $Y \overset{approx}{\sim} \mathcal{N}(100, 200)$.

(b) (13 points.)

$$\begin{aligned} \mathrm{P}(90 < Y < 95) &\approx \mathrm{P}\left(\frac{90 - 100}{\sqrt{200}} < Z < \frac{95 - 100}{\sqrt{200}}\right) \\ &= \mathrm{P}(-.707 < Z < -.354) \\ &\approx \Phi(-.35) - \Phi(-.71) \\ &= (1 - .6368) - (1 - .7611) \\ &= .1243 \end{aligned}$$

(Use the back if additional space is needed!)

Problem 5. (10 points.) Let $X = V + W$ and $Y = V + Z$, where V, W, and Z are independent and identically distributed (i.i.d.) Poisson random variables with parameter λ.

(a) Find $\mathrm{E}[X + Y]$.

(b) Find $\mathrm{Cov}(X, Y)$.

(c) Are X and Y independent? Justify your answer.

Solution:

(a) (4 points.)

$$\mathrm{E}[X + Y] = 2\mathrm{E}[V] + \mathrm{E}[W] + \mathrm{E}[Z] = 4\lambda$$

(b) (4 points.) Since V, W, Z are independent:

$$\mathrm{Cov}(X, Y) = \mathrm{Cov}(V+W, V+Z) = \mathrm{Cov}(V, V)+\mathrm{Cov}(V, Z)+\mathrm{Cov}(W, V)+\mathrm{Cov}(W, Z) = \mathrm{Var}(V) = \lambda$$

(c) (2 points.) Since X and Y have a covariance unequal to zero, they are not independent.

(Use the back if additional space is needed!)

Standard Normal Table

$\Phi(z) = P(Z \leq z)$ for $Z \sim N(0, 1)$

	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990