1. (20 pts) The following parts are unrelated.

   (a) Find $\dfrac{dy}{dx}$ for $y = \ln\left(7 - 3x^4\right)$.

   (b) Find y' for: $xe^y - \cosh(y) = 399$

Solution:

(a)

$$\frac{d}{dx}\left[\ln\left(7-3x^4\right)\right] = \frac{1}{7-3x^4}\cdot\frac{d}{dx}\left[7-3x^4\right]$$

$$= \boxed{-\frac{12x^3}{7-3x^4}}$$

(b)

$$\frac{d}{dx}\left[xe^y - \cosh(y)\right] = \frac{d}{dx}[399]$$

$$x\left(e^y \cdot y'\right) + e^y - \sinh(y)\cdot y' = 0$$

$$\left[xe^y - \sinh(y)\right]y' = -e^y$$

$$y' = \frac{-e^y}{xe^y - \sinh(y)}$$

$$y' = \boxed{-\frac{e^y}{\sinh(y) - xe^y}}$$

2. (36 pts) The following are unrelated.

   (a) Evaluate the limit (you may leave your answer in terms of hyperbolic functions): $\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^{3x}$

   (b) Evaluate the limit: $\lim_{x\to 0}\frac{\arcsin(x)}{x}$

   (c) Evaluate the definite integral $\int_1^{\ln(2)}\frac{5e^x}{e^x+1}\,dx$

   (d) Evaluate the indefinite integral $\int\frac{\sin(\theta)}{1+\cos^2(\theta)}\,d\theta$

Solution:

(a)

$$L = \lim_{x\to\infty}\left(1+\frac{2}{x}\right)^{3x}$$

$$\ln L = \ln\left[\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^{3x}\right]$$

$$\ln L = \lim_{x\to\infty}\ln\left[\left(1+\frac{2}{x}\right)^{3x}\right] \qquad \text{(natural log function is continuous)}$$

$$\ln L = \lim_{x\to\infty} 3x\ln\left(1+\frac{2}{x}\right)$$

The preceding limit is of indeterminate form of type $0\cdot\infty$ because

$$\lim_{x\to\infty}\ln\left(1+\frac{2}{x}\right) = \ln\left(1+\lim_{x\to\infty}\frac{2}{x}\right) = \ln(1+0) = \ln(1) = 0$$

So,

$$\ln L = \lim_{x\to\infty} 3x\ln\left(1+\frac{2}{x}\right)$$

$$= 3\lim_{x\to\infty}\frac{\ln(1+2/x)}{1/x}$$

The preceding limit is of indeterminate form of type $0/0$, so L'Hôpital's Rule can be applied.

$$\ln L = 3\lim_{x\to\infty}\frac{\ln(1+2/x)}{1/x}$$

$$\ln L \overset{LH}{=} 3\lim_{x\to\infty}\frac{\left(\frac{-2/x^2}{1+2/x}\right)}{(-1/x^2)}$$

$$\ln L = 3 \lim_{x\to\infty} \left(\frac{-2/x^2}{1+2/x} \right) \cdot (-x^2)$$

$$\ln L = 6 \lim_{x\to\infty} \frac{1}{1+2/x} = 6 \cdot \frac{1}{1+0} = 6$$

$$L = \boxed{e^6}$$

(b) Since $\arcsin(0) = 0$, the limit $\lim_{x\to 0} \frac{\arcsin(x)}{x}$ is of indeterminate form of type $0/0$, so L'Hôpital's Rule can be applied.

$$\lim_{x\to 0} \frac{\arcsin(x)}{x} \overset{LH}{=} \lim_{x\to 0} \frac{1/\sqrt{1-x^2}}{1}$$

$$= \frac{1}{\sqrt{1-0}} = \boxed{1}$$

(c)

$$\int_1^{\ln(2)} \frac{5e^x}{e^x+1}\, dx$$

Apply u-substitution with $u = e^x + 1$.

$$u = e^x + 1$$

$$\frac{du}{dx} = e^x$$

$$du = e^x\, dx$$

Also,

$$x = 1 \quad \Rightarrow \quad u = e + 1$$

$$x = \ln(2) \quad \Rightarrow \quad u = e^{\ln(2)} + 1 = 2 + 1 = 3$$

Therefore,

$$\int_1^{\ln(2)} \frac{5e^x}{e^x+1}\, dx = 5 \int_{e+1}^{3} \frac{du}{u}$$

$$= 5 \ln|u| \Big|_{e+1}^{3}$$

$$= 5\left(\ln(3) - \ln(e+1)\right)$$

$$= \boxed{5 \ln\left(\frac{3}{e+1}\right)}$$

(d)

$$\int \frac{\sin(\theta)}{1+\cos^2(\theta)}\, d\theta$$

Apply u-substitution with $u = \cos(\theta)$.

$$u = \cos(\theta)$$

$$\frac{du}{d\theta} = -\sin(\theta)$$

$$du = -\sin(\theta)\, d\theta$$

Therefore,

$$\begin{aligned}
\int \frac{\sin(\theta)}{1+\cos^2(\theta)}\, d\theta &= -\int \frac{1}{1+u^2}\, du \\
&= -\arctan(u) + C \\
&= \boxed{-\arctan(\cos(\theta)) + C}
\end{aligned}$$

3. (16 pts) Consider the function $f(x) = 5x^2 + 2x - 3$ on the interval $[-1, 2]$.

(a) Approximate $\int_{-1}^{2} f(x)\,dx$ using two rectangles of equal width with the right end point rule (R_2).

(b) Show that $f(x)$ satisfies the hypotheses of the Mean Value Theorem.

(c) Find all numbers, c, that satisfy the conclusion of the Mean Value Theorem.

Solution:

(a) The interval $[-1, 2]$ is to be divided into two subintervals of equal width: $[-1, 0.5]$ and $[0.5, 2]$. The width of each subinterval is $\Delta x = 1.5$.

The construction of the Riemann sum R_2 involves evaluating the given function at the righthand endpoint of each subinterval in order to obtain the heights of the approximating rectangles. Specifically,

$$\begin{aligned}
\int_{-1}^{2} f(x)\,dx \approx R_2 &= [f(0.5) + f(2)]\Delta x \\
&= \left[\left(5 \cdot 0.5^2 + 2 \cdot 0.5 - 3\right) + \left(5 \cdot 2^2 + 2 \cdot 2 - 3\right)\right] \cdot 1.5 \\
&= [(5/4 + 1 - 3) + (20 + 4 - 3)] \cdot 1.5 \\
&= (-0.75 + 21) \cdot 1.5 \\
&= \left(\frac{-3 + 84}{4}\right)\left(\frac{3}{2}\right) = \boxed{\frac{243}{8}}
\end{aligned}$$

(b) Since $f(x)$ is a polynomial, f is continuous and differentiable on $(-\infty, \infty)$. In particular, f satisfies the two hypotheses of the Mean Value Theorem on the specified interval:

$\boxed{f \text{ is contiunous on } [-1, 2]}$ and $\boxed{f \text{ is differentiable on } (-1, 2)}$

(c) A value of c on $(-1, 2)$ satisfies the conclusion of the Mean Value Theorem if $f'(c) = \dfrac{f(2) - f(-1)}{2 - (-1)}$.

$$f(2) = 5 \cdot 2^2 + 2 \cdot 2 - 3 = 20 + 4 - 1 = 21$$
$$f(-1) = 5 \cdot (-1)^2 + 2 \cdot (-1) - 3 = 5 - 2 + 3 = 0$$

Also,

$$f'(c) = 10c + 2$$

Therefore,

$$\begin{aligned}
10c + 2 &= \frac{21 - 0}{3} = 7 \\
10c &= 5 \\
c &= \boxed{1/2}
\end{aligned}$$

4. (16 pts) For $f(x) = \int_{2x}^{1} \sin^{-1}(t)\ dt$ answer the following:

(a) Find $f'(x)$.

(b) Find the equation of the line tangent to f that passes through the point $\left(\frac{1}{4}, \frac{5\pi - 6\sqrt{3}}{12}\right)$.

(c) Find $f''(x)$.

Solution:

(a)

$$\begin{aligned} f'(x) &= \frac{d}{dx}\int_{2x}^{1} \sin^{-1}(t)\ dt \\ &= -\frac{d}{dx}\int_{1}^{2x} \sin^{-1}(t)\ dt \\ &= -\sin^{-1}(2x)\cdot\frac{d}{dx}[2x] \qquad \text{(FTC-1)} \\ &= \boxed{-2\sin^{-1}(2x)} \end{aligned}$$

(b)

$$\begin{aligned} f'\left(\frac{1}{4}\right) &= -2\sin^{-1}\left(2\cdot\frac{1}{4}\right) \\ &= -2\sin^{-1}\left(\frac{1}{2}\right) \\ &= -2\cdot\frac{\pi}{6} = -\frac{\pi}{3} \end{aligned}$$

Therefore, an equation for the tangent line is $\boxed{y - \frac{5\pi - 6\sqrt{3}}{12} = -\frac{\pi}{3}\left(x - \frac{1}{4}\right)}$

(c)

$$\begin{aligned} f''(x) &= \frac{d}{dx}\left[-2\sin^{-1}(2x)\right] \\ &= -2\cdot\left(\frac{1}{\sqrt{1-(2x)^2}}\cdot\frac{d}{dx}[2x]\right) \\ &= \boxed{-\frac{4}{\sqrt{1-4x^2}}} \end{aligned}$$

5. (18 pts) The position, s, of a particle moving in a straight line is given by $s(t) = \frac{t^3}{3} - \frac{t^2}{2} - 6t$ ft for time t in seconds. The equation is valid for $t \geq 0$. Be sure to include units below where relevant.

(a) Find the velocity of the particle as a function of t.

(b) On what interval(s) of time is the particle moving in the positive direction?

(c) Find the acceleration of the particle after 3 seconds.

(d) Find the average velocity of the particle on the interval $[0, 3]$.

(e) Find the total distance traveled by the particle during the first 4 seconds.

Solution:

(a) $v(t) = \frac{d}{dt}[s(t)] = \boxed{t^2 - t - 6 \text{ ft/s}}$

(b) $v(t) = t^2 - t - 6 = (t-3)(t+2)$.

Since $s(t)$ has been defined on the domain $[0, \infty)$, the velocity $v(t)$ is defined on the domain $(0, \infty)$. The factor $(t+2)$ is positive for all values of t on $(0, \infty)$. Therefore, the velocity $v(t)$ is positive only on the interval $(3, \infty)$ so that the particle is moving in the positive direction on the interval $\boxed{(3, \infty) \text{ s}}$

(c) $a(t) = \frac{d}{dt}[v(t)] = 2t - 1$ so that $a(3) = 2 \cdot 3 - 1 = \boxed{5 \text{ ft/s}^2}$

(d)

$$\begin{aligned} v_{ave} &= \frac{1}{3-0}\int_0^3 \left(t^2 - t - 6\right) dt \\ &= \frac{1}{3}\left(\frac{t^3}{3} - \frac{t^2}{2} - 6t\right)\Big|_0^3 \\ &= \frac{1}{3}\left(\frac{3^3}{3} - \frac{3^2}{2} - 6 \cdot 3\right) \\ &= 3 - \frac{3}{2} - 6 = \boxed{-9/2 \text{ ft/s}} \end{aligned}$$

(e) The work in part (b) indicates that the particle is moving in the negative direction on $(0,3)$ and in the positive direction on $(3,4)$.

$$D = |s(3) - s(0)| + |s(4) - s(3)|$$

$$s(0) = 0$$

$$s(3) = \frac{3^3}{3} - \frac{3^2}{2} - 6 \cdot 3 = 9 - \frac{9}{2} - 18 = -\frac{27}{2}$$

$$s(4) = \frac{4^3}{3} - \frac{4^2}{2} - 6 \cdot 4 = \frac{64}{3} - 8 - 24 = \frac{64 - 96}{3} = -\frac{32}{3}$$

$$\begin{aligned} D &= \left| -\frac{27}{2} - 0 \right| + \left| -\frac{32}{3} - \left(-\frac{27}{2} \right) \right| \\ &= \frac{27}{2} + \left| -\frac{64}{6} + \frac{81}{6} \right| \\ &= \frac{81}{6} + \frac{81 - 64}{6} \\ &= \frac{162 - 64}{6} = \boxed{\frac{49}{3} \text{ ft}} \end{aligned}$$

6. (24 pts) The following parts are unrelated.

   (a) A bacteria culture initially contains 106 cells and its population, $P(t)$, grows at a rate proportional to its size. After an hour the population has increased to 420. Find a function for the number of cells after t hours.

   (b) At noon, ship A is 60 km west of ship B. Ship A is sailing south at 15 km/h and ship B is sailing north at 5 km/h. How fast is the distance between the ships changing at 4:00 PM?

Solution:

(a) Since the population grows at a rate proportional to its size, the population is undergoing exponential growth. The corresponding exponential model for $P(t)$ is as follows:

$$P(t) = P(0)e^{kt}$$

Since the initial population size is given as 106 cells, we have $P(0) = 106$, so that

$$P(t) = 106e^{kt}$$

The population size at $t = 1$ hour is given as $P(1) = 420$, so that

$$P(1) = 106e^{(k)(1)} = 106e^{k} = 420$$

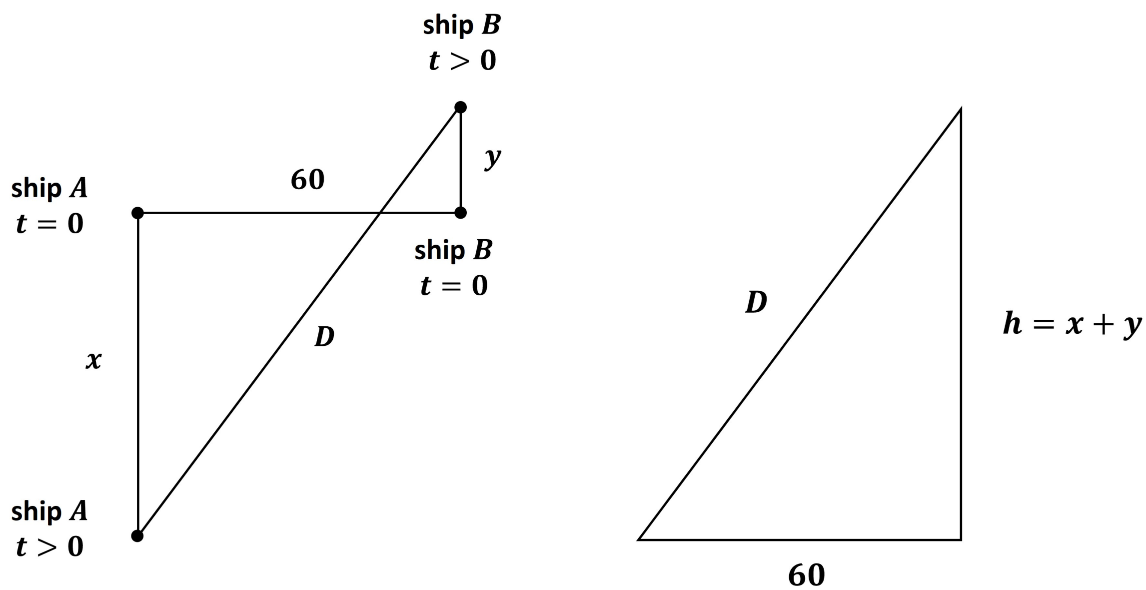
$$e^{k} = \frac{420}{106} = \frac{210}{53}$$

$$k = \ln\left(\frac{210}{53}\right)$$

Therefore, $\boxed{P(t) = 106e^{\ln\left(\frac{210}{53}\right)\cdot t} = 106 \cdot \left(\frac{210}{53}\right)^{t}}$

(b) Let $t = 0$ represent noon, so that $t = 4$ corresponds to 4:00 PM. Also, let $x = x(t)$ represent the distance ship A is from its original position at time t, let $y = y(t)$ represent the distance ship B is from its original position at time t, and let $D = D(t)$ represent the distance between the two ships at time t.

The situation is depicted in the lefthand figure below. Note that the geometrical relationship between x, y, and D can be alternatively expressed using the righthand figure below, where $h = x + y$.



Since ship A is traveling at 15 km/h, $dx/dt = 15$, and since ship B is traveling at 5 km/h, $dy/dt = 5$. Since $h = x + y$,

$$\frac{dh}{dt} = \frac{dx}{dt} + \frac{dy}{dt} = 15 + 5 = 20$$

The Pythagorean Theorem indicates that $D^2 = 60^2 + h^2$.

$$\frac{d}{dt}\left[D^2\right] = \frac{d}{dt}\left[60^2 + h^2\right]$$

$$2D\frac{dD}{dt} = 2h\frac{dh}{dt}$$

$$x(4) = (15)(4) = 60$$

$$y(4) = (5)(4) = 20$$

$$h(4) = 60 + 20 = 80$$

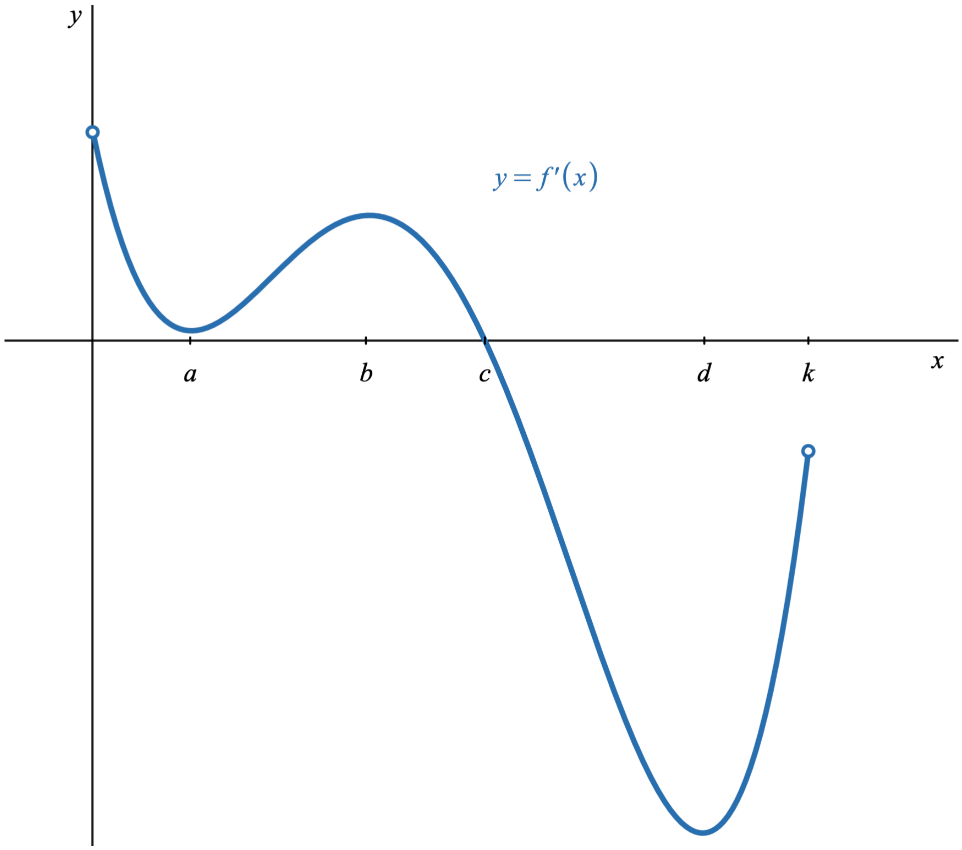
$$D(4) = \sqrt{60^2 + 80^2} = \sqrt{3600 + 6400} = 100$$

Therefore,

$$\frac{dD}{dt} = \frac{80 \cdot 20}{100} = \boxed{16 \text{ km/h}}$$

7. (20 pts) Parts (a) and (b) below are unrelated.

(a) Below is the graph of the **first derivative**, f', of a function f. Answer the following questions related to f which is defined on $(0, k)$ with x-values: a, b, c, d, and k. (List all answers that apply. Use interval notation where appropriate. No explanation is necessary.)



i. Evaluate $\lim_{h\to 0} \dfrac{f'(b+h) - f'(b)}{h}$

ii. On what intervals is f decreasing?

iii. At what x-value does f have a local maximum?

iv. On what intervals is f concave down?

v. At what value(s) of x does f have an inflection point?

(b) Sketch a graph of a **single function** $y = g(x)$ with **all** of the following properties:

- $g(0) = 1$
- $g(-x) = g(x)$
- $\lim_{x\to 1^-} g(x) = -\infty$
- $\lim_{x\to -1^-} g(x) = +\infty$
- $\lim_{x\to\infty} g(x) = 2$
- $\lim_{x\to -2} g(x) = 3$
- $g'(x) > 0$ if $x < 0$ and $x \neq -1, -2$
- $g(-2)$ DNE

Solution:

(a) i. By the definition of derivative, the given limit represents the derivative of the function $f'(x)$ with respect to x, evaluated at $x = b$. That is,

$$\lim_{h \to 0} \frac{f'(b+h) - f'(b)}{h} = \frac{d}{dx}\left[f'(x)\right]\Bigg|_{x=b} = f''(b)$$

So, the given limit represents the slope of the line tangent to the curve $y = f'(x)$, at $x = b$. From the given graph, the slope of that tangent line equals zero, since there is a horizontal tangent line there. Therefore,

$$\lim_{h \to 0} \frac{f'(b+h) - f'(b)}{h} = \boxed{0}$$

ii. The function $f(x)$ is decreasing wherever $f'(x)$ is negative. The y values of the given graph represent the values of $f'(x)$, and those y values are negative on the interval $\boxed{(c, k)}$

iii. According to the First Derivative Test, the function $f(x)$ has a local maximum value wherever the value of $f'(x)$ transitions from positive to negative. The y values of the given graph represent the values of $f'(x)$, and those y values transition from positive to negative at $x = \boxed{c}$

iv. The function $f(x)$ is concave down wherever $f''(x)$ is negative. Since $f''(x)$ is the derivative with respect to x of $f'(x)$ and the curve depicted in the given graph represents $y = f'(x)$, $f''(x)$ represents the slope of the line tangent to the given curve at x. So, $f''(x)$ is negative wherever the curve $y = f'(x)$ is decreasing. Therefore, f is concave down on $\boxed{(0, a) \cup (b, d)}$

v. The function $f(x)$ has an inflection point at any location at which f is continuous and its concavity changes. Since f' exists everywhere on the interval $(0, k)$, f is differentiable on $(0, k)$, which implies that f is continuous on that same interval.

f is concave up wherever $f'' > 0$, which occurs where $y = f'$ is increasing, and f is concave down wherever $f'' < 0$, which occurs where $y = f'(x)$ is decreasing. The given graph indicates that $y = f'(x)$ transitions from decreasing to increasing at $x = a$ and at $x = d$, and it indicates that $y = f'(x)$ transitions from increasing to decreasing at $x = b$. Therefore, f has inflection points at $x = \boxed{a, b, d}$

(b) One graph that satisfies all of the given properties is