1. (27 points) Consider $f(x,y) = x^3 + 4y^2 - 15x + 7$.

   (a) (6 points) Find all critical points of $f(x,y)$.

   (b) (6 points) Classify each of the critical points as a local maximum, local minimum, or saddle point.

   (c) (6 points) Explain why the Extreme Value Theorem guarantees there are points on the ellipse $3x^2 + 2y^2 = 48$ where $f(x,y)$ will obtain an absolute maximum value and an absolute minimum value.

   (d) (9 points) Find the absolute maximum and minimum values of $f(x,y)$ subject to the constraint $3x^2 + 2y^2 = 48$.

   Solution:

   (a) We have $f_x = 3x^2 - 15$ and $f_y = 8y$. These are simultaneously 0 only at the points $(\pm\sqrt{5}, 0)$, so these are the only critical points.

   (b) We will apply the second derivative test to $(\pm\sqrt{5}, 0)$. We have

   $$D = f_{xx}f_{yy} - f_{xy}^2 = (6x)(8) - 0^2 = 48x$$

   and $f_{xx} = 6x$. $D(\sqrt{5}, 0) > 0$ and $f_{xx}(\sqrt{5}, 0) > 0$, so $(\sqrt{5}, 0)$ is the location of a local minimum value. $D(-\sqrt{5}, 0) < 0$, so $(-\sqrt{5}, 0)$ is the location of a saddle point.

   (c) Since the ellipse is a closed and bounded set and $f(x,y)$ is continuous on the ellipse (because it is a polynomial), then the Extreme Value Theorem guarantees the existence of absolute minimum and maximum values of $f(x,y)$ over the ellipse.

   (d) We will apply Lagrange Multipliers to the ellipse, which is given by $g(x,y) = 3x^2 + 2y^2 = 48$. We need to solve the system of equations given by this constraint and $\nabla f = \lambda \nabla g$:

   $$\begin{aligned} 3x^2 - 15 &= 6\lambda x \\ 8y &= 4\lambda y \\ 3x^2 + 2y^2 &= 48. \end{aligned}$$

   The second equation tells us that $y(2 - \lambda) = 0$, so we know that $y = 0$ or $\lambda = 2$.

   Case: $y = 0$**:** The constraint reduces to $3x^2 = 48$, which yields $x = \pm 4$. Note that $f(4,0) = 11$ and $f(-4,0) = 3$.

   Case: $\lambda = 2$**:** If we apply this to the first equation, we get $3x^2 - 15 = 12x$. This quadratic equation has solutions $x = 5, -1$.

   Subcase: $x = 5$**:** The constraint becomes $75 + 2y^2 = 48$, or $2y^2 = -27$, which has no solutions.

   Subcase: $x = -1$**:** The constraint becomes $3 + 2y^2 = 48$, which has solutions $y = \pm\sqrt{\frac{45}{2}}$. We have $f\left(-1, \pm\sqrt{\frac{45}{2}}\right) = 111$.

   So, the absolute maximum value is 111, and the absolute minimum value is 3.

2. (30 points) Captain Bonaventura Cavalieri is a pirate who likes to steal acorns from unsuspecting squirrels. He recently stole an acorn from Sam the Squirrel after Sam accidentally dropped it while running in a park. Captain

Bonaventura Cavalieri stores these acorns in a vault that occupies the region in space, $\mathcal{E}$, given by $x^2 + y^2 \leq 25$, $x \leq 0$, and $0 \leq z \leq 10$.

Pam the Penguin learns of this piracy, and goes to the vault. She presses an "eject" button that jettisons all of the acorns from the vault. The movement of the acorns can be described with the velocity field

$$\mathbf{F}(x, y, z) = \langle z^2 y, x^2 y, y^2 z \rangle.$$

(a) (15 points) Setup but **do not evaluate** a double integral that gives the outward flux of the acorns through the piece of the surface of the vault given by $x^2 + y^2 = 25$ and $x \leq 0$ for $-5 \leq y \leq 5$ and $0 \leq z \leq 10$. (For full credit, this integral should be fully set-up with correct limits of integration, not be left in terms of any vectors or vector operations, and the integrand should be in terms of only the variables with which one would integrate.)

(b) (15 points) Compute the outward flux of the acorns through the entire surface of the vault. For full credit, you **will evaluate the integral** in this problem. (Hint: You may want to use one of our major vector calculus (chapter 13) theorems. If you do so, be sure to mention the name of the theorem you use.)

Solution:

(a) This surface, $\mathcal{S}$, is given by $g(x, y, z) = x^2 + y^2 = 25$. So, $\nabla g = \langle 2x, 2y, 0 \rangle$. (Note that this does point out of the surface.) We will project onto a region, $\mathcal{R}$, in the yz-plane given by $-5 \leq y \leq 5$ and $0 \leq z \leq 10$. So, $\mathbf{p} = \langle 1, 0, 0 \rangle$, and $|\nabla g \cdot \mathbf{p}| = 2|x| = -2x$ (because $x \leq 0$).
So,

$$\begin{aligned}
\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} &= \iint_{\mathcal{R}} \mathbf{F} \cdot \frac{\nabla g}{|\nabla g \cdot \mathbf{p}|} \, dA \\
&= \iint_{\mathcal{R}} -z^2 y - x y^2 \, dA \\
&= \int_0^{10} \int_{-5}^{5} -z^2 y - y^2 \sqrt{25 - y^2} \, dy \, dz.
\end{aligned}$$

(b) We will use Gauss' Divergence Theorem:

$$\begin{aligned}
\iint_{\partial \mathcal{E}} \mathbf{F} \cdot d\mathbf{S} &= \iiint_{\mathcal{E}} \text{div } \mathbf{F} \, dV \\
&= \iiint_{\mathcal{E}} x^2 + y^2 \, dV \\
&= \int_0^{10} \int_{\pi/2}^{3\pi/2} \int_0^5 r^3 \, dr \, d\theta \, dz \\
&= 10\pi \left[\frac{r^4}{4} \right]_{r=0}^{r=5} \\
&= \frac{3125\pi}{2}.
\end{aligned}$$

3. (15 points) Pam the Penguin returns an acorn to Sam the Squirrel by walking along the path, $\mathcal{C}$, that goes along the curve $x^2 + y^3 = 8$ from $(2\sqrt{2}, 0)$ to $(0, 2)$. Suppose the force required to move the acorn is given by

$$\mathbf{F}(x, y) = (y e^x + \sin y)\mathbf{i} + (e^x + x \cos y + 2y)\mathbf{j}.$$

Find the work done in moving the acorn along $\mathcal{C}$ **without parameterizing** the curve.

Solution: We need to compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ to find the work. We can observe that $\mathbf{F}$ is conservative by showing curl $\mathbf{F} = \mathbf{0}$ or by observing that

$$\frac{\partial}{\partial y}(ye^x + \sin y) = e^x + \cos y = \frac{\partial}{\partial x}(e^x + x\cos y + 2y).$$

So, the Fundamental Theorem of Line Integrals applies.

We need to find a potential function $f(x, y)$. That is, we want f such that $\mathbf{F} = \nabla f$. Since

$$\nabla f = (ye^x + \sin y)\mathbf{i} + (e^x + x\cos y + 2y)\mathbf{j},$$

then we know that $f_x = ye^x + \sin y$ and $f_y = e^x + x\cos y + 2y$. If we antidifferentiate f_x with respect to x, we obtain

$$f(x, y) = ye^x + x\sin y + g(y).$$

We can then differentiate this with respect to y and identify the result with f_y:

$$e^x + x\cos(y) + g'(y) = f_y = e^x + x\cos y + 2y.$$

It follows that $g'(y) = 2y$, so $g(y) = y^2$. (We can add an arbitrary constant, but we only need one potential function, not the family of all potential functions.) So, a potential function for $\mathbf{F}$ is

$$f(x, y) = ye^x + x\sin y + y^2.$$

By the Fundamental Theorem of Line Integrals, we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(0, 2) - f(2\sqrt{2}, 0) = (2 + 0 + 4) - (0 + 0 + 0) = 6.$$

4. (30 points) Upon the return of his long lost acorn, Sam the Squirrel is filled with excitement and starts running around in circles. Specifically, he is running counterclockwise around the circle of radius 2 centered at the origin. The force he uses in running around this circle is given by $\mathbf{F}(x, y) = \langle x^2 + y^2, x\rangle$. We will find the work Sam does in running around this circle once in two different ways:

   (a) (15 points) Parameterize the curve and compute the integral directly.

   (b) (15 points) Compute the integral by applying an appropriate major theorem from vector calculus (chapter 13). Be sure to state the name of the theorem you use.

   Solution:

   (a) We can parameterize the curve with $\mathbf{r}(t) = \langle 2\cos t, 2\sin t\rangle$, $0 \le t \le 2\pi$. Then, we have

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt \\
&= \int_0^{2\pi} \langle 4(\cos^2 t + \sin^2 t), 2\cos t\rangle \cdot \langle -2\sin t, 2\cos t\rangle\, dt \\
&= \int_0^{2\pi} -8\sin t + 4\cos^2 t\, dt \\
&= \int_0^{2\pi} -8\sin t + 2 + 2\cos(2t)t\, dt \\
&= [8\cos t + 2t + \sin(2t)]_0^{2\pi} \\
&= 4\pi.
\end{aligned}$$

(b) Let $\mathcal{D}$ be the disk bounded by this circle. Using Green's Theorem, we have

$$\begin{aligned}
\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} &= \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA \\
&= \iint_D 1 - 2y\, dA \\
&= \int_0^{2\pi} \int_0^2 (1 - 2r\sin\theta) r\, dr\, d\theta \\
&= \int_0^{2\pi} \int_0^2 r - 2r^2 \sin\theta\, dr\, d\theta \\
&= \int_0^{2\pi} \left[\frac{1}{2}r^2 - \frac{2}{3}r^3 \sin\theta\right]_0^2 d\theta \\
&= \int_0^{2\pi} 2 - \frac{16}{3}\sin\theta\, d\theta \\
&= \left[2\theta + \frac{16}{3}\cos\theta\right]_0^{2\pi} \\
&= 4\pi.
\end{aligned}$$

5. (21 points) Consider the point $P(0,0,1)$ and the plane $z = -1$.

   (a) (9 points) Determine the equation of the surface of all points that are equidistant from the point P and the plane mentioned above. Simplify the equation to a standard form presented in class for that type of surface.

   (b) (3 points) Identify the type of surface described in part (a).

   (c) (9 points) Find the equation of the plane that contains the point P and the line $\mathbf{r}(t) = 2t\mathbf{i} + (4t-3)\mathbf{j} + (2-5t)\mathbf{k}$. Write your final answer in the form $ax + by + cz = d$.

Solution:

Let (x, y, z) be a point on this surface. Then, using the distance formula we can relate the distance from this point to $(0,0,1)$ to the distance between this point and the plane $z = -1$:

$$\begin{aligned}
\sqrt{(x-0)^2 + (y-0)^2 + (z-1)^2} &= \sqrt{(x-x)^2 + (y-y)^2 + (z-(-1))^2} \\
x^2 + y^2 + z^2 - 2z + 1 &= z^2 + 2z + 1 \\
x^2 + y^2 &= 4z. \\
\left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2 &= z.
\end{aligned}$$

We have equation (a) $x^2 + y^2 = 4z$ or $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2 = z$ which represents a (b) paraboloid.

(c) We see that $(0,0,1)$ and $(0,-3,2)$ are both points on the plane, so $\langle 0, 0-(-3), 1-2\rangle = \langle 0, 3, -1\rangle$ is a vector on the plane. So, is $\langle 2, 4, -5\rangle$ the direction vector of the given line. So, a normal vector of the plane is given by

$$\langle 2, 4, -5\rangle \times \langle 0, 3, -1\rangle = \langle 11, 2, 6\rangle.$$

So, the equation of the plane becomes

$$11x + 2y + 6z = 6.$$

6. (27 points) Let $\mathcal{E}$ be the region bounded above by $x^2 + y^2 + z^2 = 17$ and below by $z^2 = 16x^2 + 16y^2$ that lies above the third quadrant of the xy-plane. Consider the integral

$$I = \iiint_{\mathcal{E}} xyz^2 \, dV.$$

(a) (6 points) Make a clear sketch of the cross-section of the solid region in the rz-plane. Assume the constant angle θ lies in the third quadrant of the xy-plane. Axes, intercepts, and curves should be clearly labeled. Shade in the region itself.

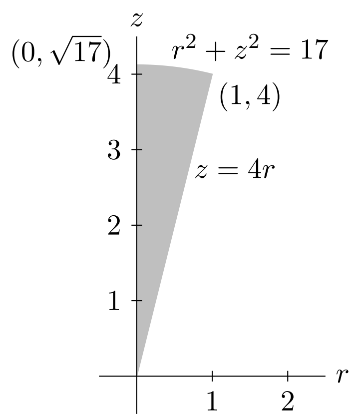
(b) (7 points) Express I as an integral or the sum of integrals in cylindrical coordinates using the order $dz\,dr\,d\theta$. Do **NOT** evaluate this integral.

(c) (7 points) Express I as an integral or the sum of integrals in spherical coordinates using the order $d\rho\,d\phi\,d\theta$. Do **NOT** evaluate this integral.

(d) (7 points) Express I as an integral or the sum of integrals in Cartesian coordinates using the order $dz\,dx\,dy$. Do **NOT** evaluate this integral.

Solution:

(a) Here is a sketch of the cross section:



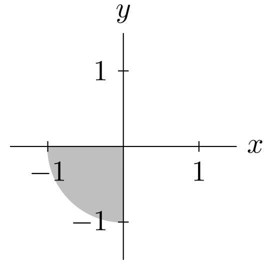
(b) Note that the third quadrant corresponds to $\pi \le \theta \le \frac{3\pi}{2}$. From that and the figure above, we have

$$I = \int_{\pi}^{3\pi/2} \int_0^1 \int_{4r}^{\sqrt{17-r^2}} r^3 \sin\theta \cos\theta z^2 \, dz\,dr\,d\theta.$$

(c) We see that the greatest value of ϕ in this region is $\phi = \arctan(1/4)$, because this is the value of ϕ that corresponds to the point $(r,z) = (1,4)$ in the figure above. Based on this and other information from the figure above, we have

$$I = \int_{\pi}^{3\pi/2} \int_0^{\arctan(1/4)} \int_0^{\sqrt{17}} \rho^6 \sin^3\phi \cos^2\phi \sin\theta\cos\theta \, d\rho\,d\phi\,d\theta.$$

(d) The projection of this solid onto the xy-plane is the following:



Based on the two figures we have drawn, we have

$$I = \int_{-1}^{0} \int_{-\sqrt{1-y^2}}^{0} \int_{4\sqrt{x^2+y^2}}^{\sqrt{17-x^2-y^2}} xyz^2 \, dz \, dx \, dy.$$