1. (34 pt) Evaluate the integral.

   (a) $\displaystyle\int \frac{12}{x^3+6x}\,dx$

   (b) $\displaystyle\int_{1/2}^{1} \frac{\sqrt{1-x^2}}{x^2}\,dx$

   (c) $\displaystyle\int_0^1 \frac{\ln x}{x^2}\,dx$ (*Hint:* first evaluate the indefinite integral)

Solution:

(a) (10 pt) Using partial fractions, we have

$$\frac{12}{x^3+6x} = \frac{12}{x\left(x^2+6\right)} = \frac{A}{x} + \frac{Bx+C}{x^2+6}.$$

Solving

$$\begin{aligned} A\left(x^2+6\right) + x(Bx+C) &= 12 \\ (A+B)x^2 + Cx + 6A &= 12 \end{aligned}$$

gives

$$\begin{aligned} A+B &= 0 \\ C &= 0 \\ 6A &= 12 \end{aligned}$$

which has the solution $A=2$, $B=-2$, $C=0$. Therefore

$$\begin{aligned} \int \frac{12}{x^3+6x}\,dx &= \int \left(\frac{2}{x} - \frac{2x}{x^2+6}\right) dx \\ &= \boxed{2\ln|x| - \ln\left(x^2+6\right) + C} \end{aligned}$$

using the substitution $u = x^2+6$, $du = 2x\,dx$ to integrate $2x/\left(x^2+6\right)$.

(b) (12 pt) Let $x = \sin\theta$, $dx = \cos\theta\,d\theta$. Then $\sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \cos\theta$ and the new bounds are $\theta = \frac{\pi}{6}$ to $\theta = \frac{\pi}{2}$.

$$\begin{aligned} \int_{1/2}^{1} \frac{\sqrt{1-x^2}}{x^2}\,dx &= \int_{\pi/6}^{\pi/2} \frac{\cos\theta}{\sin^2\theta}\cdot\cos\theta\,d\theta = \int_{\pi/6}^{\pi/2} \frac{\cos^2\theta}{\sin^2\theta}\,d\theta \\ &= \int_{\pi/6}^{\pi/2} \frac{1-\sin^2\theta}{\sin^2\theta}\,d\theta \\ &= \int_{\pi/6}^{\pi/2} \left(\csc^2\theta - 1\right)\,d\theta \\ &= \left[-\cot\theta - \theta\right]_{\pi/6}^{\pi/2} \\ &= \left(0 - \frac{\pi}{2}\right) - \left(-\sqrt{3} - \frac{\pi}{6}\right) = \boxed{\sqrt{3} - \frac{\pi}{3}} \end{aligned}$$

(c) (12 pt) Apply Integration by Parts with $u = \ln x$, $du = dx/x$, $dv = x^{-2}\,dx$, $v = -x^{-1}$.

$$\begin{aligned}\int \frac{\ln x}{x^2}\,dx &= -\frac{\ln x}{x} + \int x^{-2}\,dx \\ &= -\frac{\ln x}{x} - \frac{1}{x} + C\end{aligned}$$

The definite integral is improper.

$$\begin{aligned}\int_0^1 \frac{\ln x}{x^2}\,dx &= \lim_{t\to 0^+} \int_t^1 \frac{\ln x}{x^2} \\ &= \lim_{t\to 0^+} \left[\frac{-\ln x - 1}{x}\right]_t^1 \\ &= \lim_{t\to 0^+} \left(-1 + \frac{\ln t + 1}{t}\right) = \boxed{-\infty}\end{aligned}$$

because $\lim\limits_{t\to 0^+} \ln t = -\infty$ and so $\lim\limits_{t\to 0^+} \dfrac{\ln t}{t} = -\infty$. Therefore the integral is divergent.

2. (20 pt) Consider the region $\mathcal{R}$ in the first quadrant (Q1) bounded by $y = x^3 + 1$, $y = 1$, and $x = 1$.

   (a) Sketch and shade the region.

   (b) Set up (but <u>do not evaluate</u>) integrals to find the following quantities.

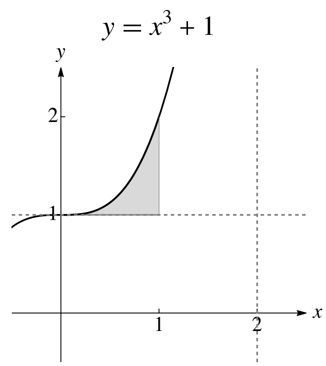
      i. Volume of the solid generated by rotating $\mathcal{R}$ about the line $x = 2$ using the Shell Method

      ii. Volume of the solid generated by rotating $\mathcal{R}$ about the line $x = 2$ using the Disk/Washer Method

      iii. Area of the surface generated by rotating the curve $y = x^3 + 1$, $0 \le x \le 1$, about the line $y = 1$

Solution:

(a) (2 pt)



(b) i. (6 pt) $V = \displaystyle\int_0^1 2\pi(2 - x)x^3\,dx$

ii. (6 pt) $V = \displaystyle\int_1^2 \pi\left[\left(2 - \sqrt[3]{y-1}\right)^2 - 1\right]\,dy$

iii. (6 pt) $S = \displaystyle\int_0^1 2\pi x^3\sqrt{1 + (3x^2)^2}\,dx$ OR $\displaystyle\int_1^2 2\pi(y-1)\sqrt{1 + \left(\frac{1}{3(y-1)^{2/3}}\right)^2}\,dy$

3. (24 pt) Determine if the following expressions converge or diverge. Justify all answers. State the names of any tests or theorems you use.

(a) $a_n = (-1)^n \dfrac{\ln(2n)}{\ln(5n)}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 - \ln n}$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}$

Solution:

(a) (8 pt)

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{\ln(2n)}{\ln(5n)} \stackrel{LH}{=} \frac{\frac{1}{2n}\cdot 2}{\frac{1}{5n}\cdot 5} = \frac{\frac{1}{n}}{\frac{1}{n}} = 1$$

As $n \to \infty$, the sequence a_n will alternate between values approaching -1 and 1, so a_n $\boxed{\text{diverges}}$.

(b) (8 pt) Use the Limit Comparison Test and compare to the convergent p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n^2-\ln n}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{1}{n^2 - \ln n} \cdot \frac{1}{\frac{1}{n^2}}$$
$$= \lim_{n\to\infty} \frac{1}{1 - \frac{\ln n}{n^2}} = \frac{1}{1-0} = 1$$

because $\displaystyle\lim_{n\to\infty} \frac{\ln n}{n^2} \stackrel{LH}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{2n} = 0.$

The limit value is $1 > 0$, so the given series also $\boxed{\text{converges}}$.

(c) (8 pt) Apply the Ratio Test.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!}{e^{(n+1)^2}} \cdot \frac{e^{n^2}}{n!}\right| = \lim_{n\to\infty} \frac{(n+1)!}{n!} \cdot \frac{e^{n^2}}{e^{n^2+2n+1}}$$
$$= \lim_{n\to\infty} \frac{n+1}{e^{2n+1}} \stackrel{LH}{=} \lim_{n\to\infty} \frac{1}{2e^{2n+1}} = 0 < 1$$

Therefore the series absolutely $\boxed{\text{converges}}$.

4. (8 pt) The nth partial sum of the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is $s_n = \dfrac{2n}{3n-1}$.

(a) Find the third term of the series.

(b) Find the sum of the series or explain why it doesn't exist.

Solution:

(a) (4 pt) Because the partial sum $s_2 = a_1 + a_2$ and $s_3 = a_1 + a_2 + a_3$, the third term is

$$a_3 = s_3 - s_2 = \frac{6}{8} - \frac{4}{5} = \boxed{-\frac{1}{20}}.$$

(b) (4 pt) The sum of the series is

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{2n}{3n-1} \overset{LH}{=} \boxed{\frac{2}{3}}.$$

5. (20 pt) Be sure to simplify your answers to the following problems.

(a) Evaluate $\int \cos\left(\sqrt{x}\right) dx$ as a power series. (*Hint:* Begin with a common Maclaurin series.)

(b) Find an approximation of $\int_0^2 \cos\left(\sqrt{x}\right) dx$ using the first 2 nonzero terms of the series found in part (a).

(c) Use the Alternating Series Estimation Theorem to find an upper bound for the approximation error. You may assume that the hypotheses of the theorem are satisfied.

Solution:

(a) (8 pt)

$$\begin{aligned}
\cos x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \\
\cos\sqrt{x} &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n}}{(2n)!} \\
\int \cos\sqrt{x}\, dx &= \int \sum_{n=0}^{\infty} (-1)^n \frac{x^{n}}{(2n)!}\, dx \\
&= \boxed{C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{(2n)!(n+1)}}
\end{aligned}$$

(b) (6 pt)

$$\begin{aligned}
\int \cos\sqrt{x}\, dx &\approx \frac{x}{0!\,1} - \frac{x^2}{2!\,2} \\
&= x - \frac{x^2}{4} \\
\int_0^2 \cos\sqrt{x}\, dx &\approx \left[x - \frac{x^2}{4}\right]_0^2 \\
&= \left(2 - \frac{2^2}{4}\right) - 0 = \boxed{1}
\end{aligned}$$

(c) (6 pt) By ASET, an upper bound for the approximation error is the next term of the series $\frac{2^3}{4!\,3} = \boxed{\frac{1}{9}}$.

6. (18 pt) Consider the parametric curve $x = 2\cos t$, $y = 1 + \sin t$ for $0 \le t \le 2\pi$.

   (a) Find a Cartesian equation of the curve. Fully simplify your answer.

   (b) Sketch the parametric curve. Indicate with an arrow the direction in which the curve is traced as t increases.

   (c) Find the slope of the line tangent to the curve at $t = \pi/4$.

Solution:

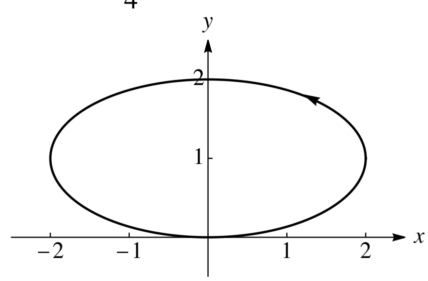
(a) (6 pt)

$$\cos^2 t + \sin^2 t = 1$$

$$\boxed{\frac{x^2}{4} + (y-1)^2 = 1}$$

(b) (6 pt)

$$\frac{x^2}{4} + (y-1)^2 = 1$$



(c) (6 pt)

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos t}{-2\sin t}$$

$$\left.\frac{dy}{dx}\right|_{t=\pi/4} = -\frac{\cos\frac{\pi}{4}}{2\sin\frac{\pi}{4}} = -\frac{\frac{1}{\sqrt{2}}}{2\cdot\frac{1}{\sqrt{2}}} = \boxed{-\frac{1}{2}}$$

7. (26 pt) Consider the polar curves $r_1 = \cos(3\theta)$ and $r_2 = \frac{1}{2}$.

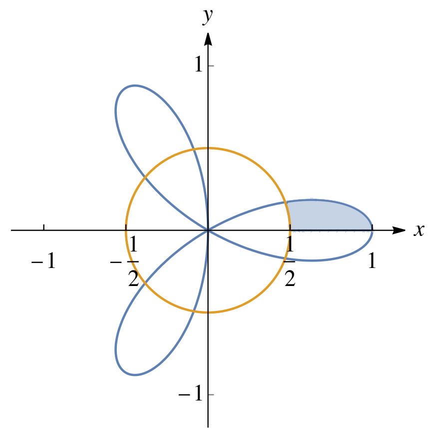
   (a) Evaluate an integral to find the area of the region in the first quadrant (Q1) inside r_1 and outside r_2.

   (b) Set up (but do not evaluate) integrals to find the total length of the perimeter (boundary) of the region described in part (a).

   (c) Use the identity $\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$ to find a Cartesian equation of the curve $r = \cos(3\theta)$. It is not necessary to simplify or to solve for y explicitly.

Solution:

(a) (12 pt)



In Q1, the curves intersect at $\cos(3\theta) = \frac{1}{2} \implies 3\theta = \frac{\pi}{3} \implies \theta = \frac{\pi}{9}$.

$$\begin{aligned}
A &= \int_{\alpha}^{\beta} \frac{1}{2}\left(r_1^2 - r_2^2\right) d\theta \\
&= \int_0^{\pi/9} \frac{1}{2}\left(\cos^2(3\theta) - \frac{1}{4}\right) d\theta \\
&= \int_0^{\pi/9} \frac{1}{2}\cos^2(3\theta) d\theta - \int_0^{\pi/9} \frac{1}{8} d\theta \\
&= \int_0^{\pi/9} \frac{1}{4}(1 + \cos(6\theta))\, d\theta - \int_0^{\pi/9} \frac{1}{8} d\theta \\
&= \left[\frac{1}{4}\left(\theta + \frac{1}{6}\sin(6\theta)\right)\right]_0^{\pi/9} - \left[\frac{1}{8}\theta\right]_0^{\pi/9} \\
&= \frac{1}{4}\left(\frac{\pi}{9} + \frac{1}{6}\cdot\frac{\sqrt{3}}{2}\right) - \frac{1}{8}\cdot\frac{\pi}{9} \\
&= \boxed{\frac{\pi}{72} + \frac{\sqrt{3}}{48}}
\end{aligned}$$

(b) (8 pt) The perimeter is formed by r_1 and r_2 for $0 \le \theta \le \frac{\pi}{9}$, and a line segment of length $\frac{1}{2}$.

$$\begin{aligned}
L &= \frac{1}{2} + \int_0^{\pi/9} \sqrt{r_1^2 + \left(\frac{dr_1}{d\theta}\right)^2}\, d\theta + \int_0^{\pi/9} \sqrt{r_2^2 + \left(\frac{dr_2}{d\theta}\right)^2}\, d\theta \\
&= \boxed{\frac{1}{2} + \int_0^{\pi/9} \sqrt{\cos^2(3\theta) + (-3\sin(3\theta))^2}\, d\theta + \int_0^{\pi/9} \sqrt{\frac{1}{4}}\, d\theta} \\
&= \frac{1}{2} + \int_0^{\pi/9} \sqrt{1 + 8\sin^2(3\theta)}\, d\theta + \frac{\pi}{18}
\end{aligned}$$

(c) (6 pt) Use the given identity and the identities $r^2 = x^2 + y^2$ and $x = r\cos\theta$.

$$\begin{aligned} r &= \cos(3\theta) \\ r &= 4\cos^3\theta - 3\cos\theta \end{aligned}$$

Multiply both sides by r^3, then substitute.

$$r^4 = 4r^3\cos^3\theta - 3r^2 \cdot r\cos\theta$$

$$\boxed{\left(x^2 + y^2\right)^2 = 4x^3 - 3x\left(x^2 + y^2\right)}$$

Alternate Solution: If $r^2 = x^2 + y^2 \neq 0$, then $\cos\theta = x/r$ and

$$\sqrt{x^2 + y^2} = \frac{4x^3}{(x^2 + y^2)^{3/2}} - \frac{3x}{\sqrt{x^2 + y^2}}.$$