1. [2360/050725 (24 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given. Please write your answers in a single column separate from any work you do to arrive at the answer.

   (a) If $\mathbf{G}$ and $\mathbf{H}$ are nonsingular matrices, then $\left(\mathbf{G}^2\mathbf{H}^{\mathrm{T}}\right)^{-1} = \left(\mathbf{H}^{-1}\right)^{\mathrm{T}}\left(\mathbf{G}^{-1}\right)^2$.

   (b) The equation $y' = 2y\left(1 - \dfrac{y}{2}\right) + 3$ has an unstable equilibrium at $y = -1$.

   (c) The set of functions $\{\sin t\cos t, \sin 2t\}$ is linearly independent on the real line.

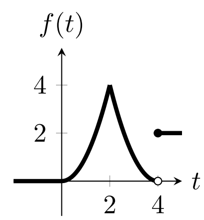
   (d) There exist solutions to $y' = t^4(1+y^2)$ that approach 0 as $t \to \infty$.

   (e) Picard's Theorem guarantees that the initial value problem $y' = \dfrac{t}{2-y}, y(1) = 2$ has no solution.

   (f) The set, $\mathbb{W}$, of vectors $[x_1\ x_2\ x_3]^{\mathrm{T}}$ in $\mathbb{R}^3$, where $x_1 + x_2 = x_3$, is a subspace of $\mathbb{R}^3$.

   (g) If $f(x,y) \geq 0$ and $g(x,y) > 0$ for all x, y, then the system $\begin{array}{l} x' = f(x,y) \\ y' = g(x,y) \end{array}$ has an equilibrium solution at the origin.

   (h) The following figure is the graph of $f(t) = t^2\left[\text{step}(t) - \text{step}(t-2)\right] + (t-4)^2\,\text{step}(t-2) + \left[2 - (t-4)^2\right]\text{step}(t-4)$.



<u>**SOLUTION:**</u>

   (a) **TRUE** $\left(\mathbf{G}^2\mathbf{H}^{\mathrm{T}}\right)^{-1} = \left(\mathbf{H}^{\mathrm{T}}\right)^{-1}\left(\mathbf{G}^2\right)^{-1} = \left(\mathbf{H}^{-1}\right)^{\mathrm{T}}(\mathbf{G}\mathbf{G})^{-1} = \left(\mathbf{H}^{-1}\right)^{\mathrm{T}}\mathbf{G}^{-1}\mathbf{G}^{-1} = \left(\mathbf{H}^{-1}\right)^{\mathrm{T}}\left(\mathbf{G}^{-1}\right)^2$

   (b) **TRUE** $y' = 2y\left(1 - \dfrac{y}{2}\right) + 3 = 2y - y^2 + 3 = -(y-3)(y+1)$. If $y < -1, y' < 0$ and if $3 > y > -1, y' > 0 \implies y = -1$ is an unstable equilibrium.

   (c) **FALSE** Rearranging the trig identity $\sin 2t = 2\sin t\cos t$ we have $\sin t\cos t - \frac{1}{2}\sin 2t = 0$, which holds for all $t \in \mathbb{R}$, indicating that the functions are linearly dependent. Note that the Wronskian of the functions vanishes so that the Wronskian test is inconclusive.

   (d) **FALSE** Since $y' > 0$ for all t and y, all solutions grow without bound as $t \to \infty$ regardless of the initial conditions.

   (e) **FALSE** Picard's theorem guarantees nothing since $\dfrac{t}{2-y}$ is not defined at $(1,2)$ and therefore is not continuous in a rectangle surrounding $(1,2)$.

   (f) **TRUE** Let $\vec{\mathbf{u}} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \vec{\mathbf{v}} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbb{W}$. Then $u_1 + u_2 = u_3$ and $v_1 + v_2 = v_3$. With $a, b \in \mathbb{R}$ we have

$$a\vec{\mathbf{u}} + b\vec{\mathbf{v}} = a\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + b\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} au_1 + bv_1 \\ au_2 + bv_2 \\ au_3 + bv_3 \end{bmatrix}$$

   with $(au_1 + bv_1) + (au_2 + bv_2) = a(u_1 + u_2) + b(v_1 + v_2) = au_3 + bv_3 \implies a\vec{\mathbf{u}} + b\vec{\mathbf{v}} \in \mathbb{W}$.

   (g) **FALSE** Since $g(x,y) > 0$, the system has no h nullclines and therefore can possess no equilibrium points.

   (h) **TRUE** $f(t) = \begin{cases} 0 & t < 0 \\ t^2 & 0 \leq t < 2 \\ (t-4)^2 & 2 \leq t < 4 \\ 2 & 4 \leq t \end{cases}$

2. [2360/050725 (8 pts)] Find the inverse Laplace transform of $Z(s) = \dfrac{e^{-2s}(s-4)}{s^2 - 8s + 41}$.

SOLUTION:

$$\mathscr{L}^{-1}\left\{\frac{e^{-2s}(s-4)}{s^2-8s+41}\right\} = \mathscr{L}^{-1}\left\{\frac{e^{-2s}(s-4)}{(s-4)^2+25}\right\}$$
$$= \text{step}\,(t-2)\left(e^{4t}\mathscr{L}^{-1}\left\{\frac{s}{s^2+25}\bigg|_{s\to s-4}\right\}\right)\Bigg|_{t\to t-2}$$
$$= \text{step}\,(t-2)\left(e^{4t}\cos 5t\right)\Big|_{t\to t-2}$$
$$= e^{4t-8}\cos(5t-10)\,\text{step}\,(t-2)$$

■

3. [2360/050725 (8 pts)] Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & k & 1 \\ 2 & -1 & k \end{bmatrix}$ where k is a real constant.

(a) (6 pts) For what value(s) of k will Col $\mathbf{A} = \mathbb{R}^3$?

(b) (2 pts) For what value(s) of k will the linear system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{0}}$ be consistent?

SOLUTION:

(a) This is equivalent to saying that $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ is consistent for all $\vec{\mathbf{b}} \in \mathbb{R}^3$. This will occur if and only if $|\mathbf{A}| \neq 0$.

$$|\mathbf{A}| = \begin{vmatrix} 1 & -1 & 0 \\ 0 & k & 1 \\ 2 & -1 & k \end{vmatrix} = 1(-1)^{1+1}\begin{vmatrix} k & 1 \\ -1 & k \end{vmatrix} - 1(-1)^{1+2}\begin{vmatrix} 0 & 1 \\ 2 & k \end{vmatrix} = k^2+1-2 = k^2-1$$

Therefore, Col $\mathbf{A} = \mathbb{R}^3$ if $\boxed{k \neq \pm 1}$.

(b) Homogeneous systems are always consistent, regardless of the value of $|\mathbf{A}|$. Thus $\boxed{k \in \mathbb{R}}$

■

4. [2360/050725 (15 pts)] Consider the equation $r^3(r-1)\left(r^2-6r+13\right) = 0$.

(a) (4 pts) Write a differential equation having the above equation as its characteristic equation.

(b) (6 pts) Find a basis of the solution space of the differential equation from part (a) and state the dimension of the solution space.

(c) (5 pts) If the differential equation from part (a) has the forcing function $f(t) = 6 + te^t + \sin 2t$ and you are solving the nonhomogeneous equation using the Method of Undetermined Coefficients, write the appropriate guess for the particular solution. Do **not** solve for the coefficients.

SOLUTION:

(a) Since

$$r^3(r-1)\left(r^2-6r+13\right) = \left(r^4-r^3\right)\left(r^2-6r+13\right)$$
$$= r^6-6r^5+13r^4-r^5+6r^4-13r^3$$
$$= r^6-7r^5+19r^4-13r^3$$

the differential equation having the given characteristic equation is $y^{(6)} - 7y^{(5)} + 19y^{(4)} - 13y^{(3)} = 0$

(b) The roots of the quadratic factor in the equation are $r = \dfrac{6 \pm \sqrt{(-6)^2-4(13)}}{2} = 3 \pm 2i$. The other roots are $r = 1$, and $r = 0$ with multiplicity 3. Thus, a basis for the solution space is $\left\{1, t, t^2, e^t, e^{3t}\sin 2t, e^{3t}\cos 2t\right\}$. The dimension of the solution space is 6.

(c) $y_p = At^3 + (Bt^2 + Ct)e^t + D\sin 2t + E\cos 2t$

■

5. [2360/050725 (20 pts)] Consider the variable coefficient linear homogeneous system $t\vec{\mathbf{x}}' = \begin{bmatrix} -1 & 6 \\ 2 & -2 \end{bmatrix}\vec{\mathbf{x}}$, $t > 0$ with $\vec{\mathbf{x}}(1) = \begin{bmatrix} -8 \\ 3 \end{bmatrix}$. Like in the written homework, assuming solutions of the form $\vec{\mathbf{x}} = t^\lambda\vec{\mathbf{v}}$, where $\lambda, \vec{\mathbf{v}}$ is an eigenvalue/eigenvector pair of the given matrix, use techniques similar to those used to construct solutions to the constant coefficient linear homogeneous systems to solve

the initial value problem. **Use Cramer's Rule** to solve the system resulting from applying the initial conditions and write your final answer as a single vector.

SOLUTION:

$$|\mathbf{A}-\lambda\mathbf{I}| = \begin{vmatrix} -1-\lambda & 6 \\ 2 & -2-\lambda \end{vmatrix} = (-1-\lambda)(-2-\lambda)-12 = \lambda^2+3\lambda-10 = (\lambda+5)(\lambda-2) = 0 \implies \lambda = -5, 2$$

$$\lambda_1 = 2 : (\mathbf{A}-2\mathbf{I})\vec{\mathbf{v}}_1 = \vec{\mathbf{0}} \to \left[\begin{array}{cc|c} -3 & 6 & 0 \\ 2 & -4 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right] \implies \vec{\mathbf{v}}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\lambda_2 = -5 : (\mathbf{A}+5\mathbf{I})\vec{\mathbf{v}}_2 = \vec{\mathbf{0}} \to \left[\begin{array}{cc|c} 4 & 6 & 0 \\ 2 & 3 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & \frac{3}{2} & 0 \\ 0 & 0 & 0 \end{array}\right] \implies \vec{\mathbf{v}}_2 = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$$\vec{\mathbf{x}}(t) = c_1 t^2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 t^{-5} \begin{bmatrix} -3 \\ 2 \end{bmatrix} \quad \text{apply initial condition} \quad \vec{\mathbf{x}}(1) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -8 \\ 3 \end{bmatrix} \implies \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -8 \\ 3 \end{bmatrix}$$

$$c_1 = \frac{\begin{vmatrix} -8 & -3 \\ 3 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix}} = \frac{-7}{7} = -1 \qquad c_2 = \frac{\begin{vmatrix} 2 & -8 \\ 1 & 3 \end{vmatrix}}{\begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix}} = \frac{14}{7} = 2$$

$$\vec{\mathbf{x}}(t) = -t^2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2t^{-5} \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -2t^2 - 6t^{-5} \\ -t^2 + 4t^{-5} \end{bmatrix}$$

■

6. [2360/050725 (18 pts)] Let $\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ -3 & 2 & 1 \end{bmatrix}$, $\mathbf{U} = \begin{bmatrix} -1 & 2 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & -1 \end{bmatrix}$ and $\vec{\mathbf{b}} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$.

(a) (4 pts) Without using any elementary row operations, show that $\mathbf{L}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$.

(b) (4 pts) Compute $\mathbf{LU}$.

(c) (10 pts) Solve $\mathbf{U}\vec{\mathbf{x}} = \mathbf{L}^{-1}\vec{\mathbf{b}}$ by finding and applying an appropriate inverse matrix. **Use Gauss-Jordan elimination** to find the inverse.

SOLUTION:

(a) We need to show that $\mathbf{LL}^{-1} = \mathbf{I}$ or $\mathbf{L}^{-1}\mathbf{L} = \mathbf{I}$

$$\mathbf{LL}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ -3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}$$

(b)

$$\mathbf{LU} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ -3 & 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 2 \\ -1 & 1 & -2 \\ 3 & -4 & 1 \end{bmatrix}$$

(c) We need to find the inverse of $\mathbf{U}$.

$$\left[\begin{array}{ccc|ccc} -1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} -1 & 2 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} -1 & 0 & 0 & 1 & -2 & -6 \\ 0 & 1 & 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 2 & 6 \\ 0 & 1 & 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right] \implies \mathbf{U}^{-1} = \begin{bmatrix} -1 & 2 & 6 \\ 0 & 1 & 4 \\ 0 & 0 & -1 \end{bmatrix}$$

Then

$$\vec{\mathbf{x}} = \mathbf{U}^{-1}\mathbf{L}^{-1}\vec{\mathbf{b}} = \begin{bmatrix} -1 & 2 & 6 \\ 0 & 1 & 4 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 6 \\ 0 & 1 & 4 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 30 \\ 21 \\ -5 \end{bmatrix}$$

7. [2360/050725 (13 pts)] Consider the system of equations $\begin{aligned} x_1 + \quad x_2 - 4x_3 + 4x_4 &= 1 \\ -2x_2 + 6x_3 \qquad &= 4 \end{aligned}$.

(a) (8 pts) Find a basis for the solution space of the associated homogeneous problem and state the dimension of the solution space.

(b) (4 pts) Write the general solution of the system.

(c) (1 pt) What is the rank of the coefficient matrix?

SOLUTION:

All parts of the problem depend on finding the RREF of the augmented matrix.

$$\left[\begin{array}{cccc|c} 1 & 1 & -4 & 4 & 1 \\ 0 & -2 & 6 & 0 & 4 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & -4 & 4 & 1 \\ 0 & 1 & -3 & 0 & -2 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & -1 & 4 & 3 \\ 0 & 1 & -3 & 0 & -2 \end{array}\right]$$

(a) Since x_3 and x_4 are free variables, solutions to the homogeneous system have the form

$$\vec{\mathbf{x}}_h = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} r - 4s \\ 3r \\ r \\ s \end{bmatrix} = r \begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \ r, s \in \mathbb{R}$$

so a basis for the solution space of the associated homogeneous problem is $\left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ which has dimension 2.

(b) Particular solutions will have the form $\vec{\mathbf{x}}_p = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 + r - 4s \\ -2 + 3r \\ r \\ s \end{bmatrix}$ and choosing $r = s = 0$ a particular solution is

$$\vec{\mathbf{x}}_p = \begin{bmatrix} 3 \\ -2 \\ 0 \\ 0 \end{bmatrix}$$

The general solution to the original system is then

$$\vec{\mathbf{x}} = \vec{\mathbf{x}}_h + \vec{\mathbf{x}}_p = r \begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ -2 \\ 0 \\ 0 \end{bmatrix}, \ r, s \in \mathbb{R}$$

(c) Since there are two pivot columns in the coefficient matrix, its rank is 2.

8. [2360/050725 (16 pts)] An harmonic oscillator is governed by the differential equation $\ddot{x} + 2\dot{x} + x = 2\delta(t-1) + 50\sin 3t$. The following identity may be helpful:

$$\frac{50}{(x^2+9)(x+1)^2} = \frac{1}{x+1} + \frac{5}{(x+1)^2} - \frac{x+4}{x^2+9}$$

(a) (2 pts) Is the oscillator undamped? critically damped? overdamped? underdamped? Justify your answer.

(b) (12 pts) Assuming that the mass starts from rest at its equilibrium position, find the equation of motion of the oscillator.

(c) (2 pts) Find the amplitude of the steady state motion.

SOLUTION:

(a) $b^2 - 4mk = 2^2 - 4(1)(1) = 0$ meaning the oscillator is critically damped.

(b) Taking Laplace transforms gives

$$s^2X(s) - sx(0) - \dot{x}(0) + 2[sX(s) - x(0)] + X(s) = 2e^{-s} + \frac{50(3)}{s^2+9}$$

$$\left(s^2+2s+1\right)X(s) = 2e^{-s} + \frac{50(3)}{s^2+9}$$

$$X(s) = \frac{2e^{-s}}{(s+1)^2} + \frac{50(3)}{(s^2+9)(s+1)^2}$$

$$X(s) = \frac{2e^{-s}}{(s+1)^2} + 3\left[\frac{1}{s+1} + \frac{5}{(s+1)^2} - \frac{s+4}{s^2+9}\right]$$

$$x(t) = \mathscr{L}^{-1}\left\{\frac{2e^{-s}}{(s+1)^2} + \frac{3}{s+1} + \frac{15}{(s+1)^2} - \frac{3s}{s^2+9} - 4\frac{3}{s^2+9}\right\}$$

$$= 2\,\text{step}\,(t-1)\left[e^{-t}\mathscr{L}^{-1}\left\{\frac{1}{s^2}\Big|_{s\to s+1}\right\}\right]\Bigg|_{t\to t-1} + 3e^{-t} + 15\left[e^{-t}\mathscr{L}^{-1}\left\{\frac{1}{s^2}\Big|_{s\to s+1}\right\}\right] - 3\cos 3t - 4\sin 3t$$

$$= 2te^{-t}\Big|_{t\to t-1}\text{step}(t-1) + 3e^{-t} + 15te^{-t} - 3\cos 3t - 4\sin 3t$$

$$= 2(t-1)e^{-(t-1)}\,\text{step}(t-1) + 3e^{-t} + 15te^{-t} - 3\cos 3t - 4\sin 3t$$

(c) The steady state solution is $x_{ss}(t) = -3\cos 3t - 4\sin 3t$ the amplitude of which is $\sqrt{(-3)^2+(-4)^2} = 5$.

■

9. [2360/050725 (13 pts)] An object with temperature $T(t)$ is located in a room whose constant temperature is 20. When $t=0$, the object's temperature is 40 and when $t=\pi/2$ its temperature is $20+20e^2$. The object is covered with a magic blanket such that the differential equation governing the object's temperature is a modified version of Newton's Law of Cooling given by $T' = (a\cos t)(T-20)$ where a is a constant to be found.

(a) (3 pts) Is the equation linear? homogeneous? separable?

(b) (10 pts) **Use the integrating factor method** to find the explicit function giving the object's temperature as a function of time.

SOLUTION:

(a) linear (yes); homogeneous (no), separable (yes)

(b)

$$T' - (a\cos t)T = -20a\cos t \implies p(t) = -a\cos t \implies \int p(t)\,\mathrm{d}t = -a\sin t \implies \mu(t) = e^{-a\sin t}$$

$$\int \left(e^{-a\sin t}T\right)'\,\mathrm{d}t = \int -20a\cos t e^{-a\sin t}\,\mathrm{d}t \qquad u = -a\sin t$$

$$e^{-a\sin t}T = 20e^{-a\sin t} + C$$

$$T(t) = 20 + Ce^{a\sin t}$$

$$T(0) = 40 = 20 + C \implies C = 20 \implies T(t) = 20\left(1+e^{a\sin t}\right)$$

$$T(\pi/2) = 20 + 20e^2 = 20\left(1+e^a\right) \implies a = 2$$

$$T(t) = 20\left(1+e^{2\sin t}\right)$$

■

10. [2360/050725 (15 pts)] Consider the system $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}}$ where $\mathbf{A} = \begin{bmatrix} a & 2 \\ 0 & 1 \end{bmatrix}$. Find all real values of a, if any, such that the system possesses the following stability and/or geometry properties. Recall that fixed points, critical points, equilibrium points and equilibrium solutions all refer to the same thing, that is, these are vectors which make $\vec{\mathbf{x}}' = \vec{\mathbf{0}}$. No partial credit available.

(a) There exist nonisolated fixed points.

(b) The isolated fixed point at $(0,0)$ is a center.

(c) Any fixed point(s) is(are) stable.

(d) The isolated fixed point at $(0, 0)$ is a saddle.

(e) The isolated fixed point at $(0, 0)$ is an unstable degenerate or star node.

SOLUTION:

Begin by noting that $\text{Tr}\,\mathbf{A} = a + 1$, $|\mathbf{A}| = a$, $(\text{Tr}\,\mathbf{A})^2 - 4|\mathbf{A}| = (a+1)^2 - 4a = a^2 + 2a + 1 - 4a = a^2 - 2a + 1 = (a-1)^2$

(a) Nonisolated fixed points exist when $|\mathbf{A}| = 0$. $\boxed{a = 0}$

(b) For a center, we need $\text{Tr}\,\mathbf{A} = 0$, which occurs for $a = -1$. However, since $|\mathbf{A}| = a$, this forces $|\mathbf{A}| < 0$ which is a saddle. $\boxed{\text{None}}$

(c) To be stable requires $\text{Tr}\,\mathbf{A} \leq 0 \implies a + 1 \leq 0 \implies a \leq -1$. Additionally, we must have $|\mathbf{A}| \geq 0 \implies a \geq 0$. These two conditions cannot be satisfied simultaneously. $\boxed{\text{None}}$

(d) We need $|\mathbf{A}| < 0$. $\boxed{a < 0}$

(e) We need $\text{Tr}\mathbf{A} > 0 \implies a + 1 > 0 \implies a > -1$ and $(\text{Tr}\,\mathbf{A})^2 - 4|\mathbf{A}| = 0 \implies (a-1)^2 = 0 \implies a = 1$. $\boxed{a = 1}$.

■