- This exam is worth 150 points and has 10 problems.
- Show all work and simplify your answers! Answers with no justification will receive no points unless otherwise noted.
- Begin each problem on a new page.
- **DO NOT LEAVE THE EXAM UNTIL YOUR HAVE SATISFACTORILY SCANNED AND UPLOADED YOUR EXAM TO GRADESCOPE.**
- You are taking this exam in a proctored and honor code enforced environment. No calculators, cell phones, or other electronic devices or the internet are permitted during the exam. You are allowed one 8.5"× 11" crib sheet with writing on two sides.

0. At the top of the first page that you will be scanning and uploading to Gradescope, write the following statement and sign your name to it: "I will abide by the CU Boulder Honor Code on this exam." **FAILURE TO INCLUDE THIS STATEMENT AND YOUR SIGNATURE MAY RESULT IN A PENALTY.**

1. [2360/050725 (24 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given. Please write your answers in a single column separate from any work you do to arrive at the answer.

   (a) If $\mathbf{G}$ and $\mathbf{H}$ are nonsingular matrices, then $\left(\mathbf{G}^2\mathbf{H}^{\mathrm{T}}\right)^{-1} = \left(\mathbf{H}^{-1}\right)^{\mathrm{T}}\left(\mathbf{G}^{-1}\right)^2$.

   (b) The equation $y' = 2y\left(1-\dfrac{y}{2}\right)+3$ has an unstable equilibrium at $y=-1$.

   (c) The set of functions $\{\sin t\cos t, \sin 2t\}$ is linearly independent on the real line.

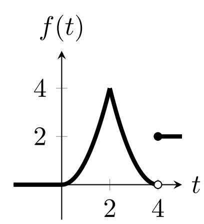
   (d) There exist solutions to $y' = t^4(1+y^2)$ that approach 0 as $t\to\infty$.

   (e) Picard's Theorem guarantees that the initial value problem $y' = \dfrac{t}{2-y}, y(1)=2$ has no solution.

   (f) The set, $\mathbb{W}$, of vectors $[x_1\ x_2\ x_3]^{\mathrm{T}}$ in $\mathbb{R}^3$, where $x_1+x_2=x_3$, is a subspace of $\mathbb{R}^3$.

   (g) If $f(x,y)\geq 0$ and $g(x,y)>0$ for all x,y, then the system $\begin{array}{l} x' = f(x,y) \\ y' = g(x,y)\end{array}$ has an equilibrium solution at the origin.

   (h) The following figure is the graph of $f(t) = t^2\left[\text{step}(t)-\text{step}(t-2)\right] + (t-4)^2\,\text{step}(t-2) + \left[2-(t-4)^2\right]\text{step}(t-4)$.



2. [2360/050725 (8 pts)] Find the inverse Laplace transform of $Z(s) = \dfrac{e^{-2s}(s-4)}{s^2-8s+41}$.

3. [2360/050725 (8 pts)] Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & k & 1 \\ 2 & -1 & k\end{bmatrix}$ where k is a real constant.

   (a) (6 pts) For what value(s) of k will Col $\mathbf{A} = \mathbb{R}^3$?

   (b) (2 pts) For what value(s) of k will the linear system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{0}}$ be consistent?

4. [2360/050725 (15 pts)] Consider the equation $r^3(r-1)\left(r^2-6r+13\right)=0$.

   (a) (4 pts) Write a differential equation having the above equation as its characteristic equation.

   (b) (6 pts) Find a basis of the solution space of the differential equation from part (a) and state the dimension of the solution space.

   (c) (5 pts) If the differential equation from part (a) has the forcing function $f(t) = 6+te^t+\sin 2t$ and you are solving the nonhomogeneous equation using the Method of Undetermined Coefficients, write the appropriate guess for the particular solution. Do **not** solve for the coefficients.

5. [2360/050725 (20 pts)] Consider the variable coefficient linear homogeneous system $t\vec{\mathbf{x}}' = \begin{bmatrix} -1 & 6 \\ 2 & -2\end{bmatrix}\vec{\mathbf{x}},\ t>0$ with $\vec{\mathbf{x}}(1) = \begin{bmatrix} -8 \\ 3\end{bmatrix}$.

   Like in the written homework, assuming solutions of the form $\vec{\mathbf{x}} = t^{\lambda}\vec{\mathbf{v}}$, where $\lambda, \vec{\mathbf{v}}$ is an eigenvalue/eigenvector pair of the given matrix, use techniques similar to those used to construct solutions to the constant coefficient linear homogeneous systems to solve the initial value problem. **Use Cramer's Rule** to solve the system resulting from applying the initial conditions and write your final answer as a single vector.

MORE PROBLEMS AND LAPLACE TRANSFORM TABLE BELOW/ON REVERSE

6. [2360/050725 (18 pts)] Let $\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ -3 & 2 & 1 \end{bmatrix}$, $\mathbf{U} = \begin{bmatrix} -1 & 2 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & -1 \end{bmatrix}$ and $\vec{\mathbf{b}} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$.

   (a) (4 pts) Without using any elementary row operations, show that $\mathbf{L}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$.

   (b) (4 pts) Compute $\mathbf{LU}$.

   (c) (10 pts) Solve $\mathbf{U}\vec{\mathbf{x}} = \mathbf{L}^{-1}\vec{\mathbf{b}}$ by finding and applying an appropriate inverse matrix. **Use Gauss-Jordan elimination** to find the inverse.

7. [2360/050725 (13 pts)] Consider the system of equations $\begin{aligned} x_1 + \quad x_2 - 4x_3 + 4x_4 &= 1 \\ -2x_2 + 6x_3 \qquad\quad &= 4 \end{aligned}$.

   (a) (8 pts) Find a basis for the solution space of the associated homogeneous problem and state the dimension of the solution space.

   (b) (4 pts) Write the general solution of the system.

   (c) (1 pt) What is the rank of the coefficient matrix?

8. [2360/050725 (16 pts)] An harmonic oscillator is governed by the differential equation $\ddot{x} + 2\dot{x} + x = 2\delta(t-1) + 50\sin 3t$. The following identity may be helpful:
$$\frac{50}{(x^2+9)(x+1)^2} = \frac{1}{x+1} + \frac{5}{(x+1)^2} - \frac{x+4}{x^2+9}$$

   (a) (2 pts) Is the oscillator undamped? critically damped? overdamped? underdamped? Justify your answer.

   (b) (12 pts) Assuming that the mass starts from rest at its equilibrium position, find the equation of motion of the oscillator.

   (c) (2 pts) Find the amplitude of the steady state motion.

9. [2360/050725 (13 pts)] An object with temperature $T(t)$ is located in a room whose constant temperature is 20. When $t = 0$, the object's temperature is 40 and when $t = \pi/2$ its temperature is $20 + 20e^2$. The object is covered with a magic blanket such that the differential equation governing the object's temperature is a modified version of Newton's Law of Cooling given by $T' = (a\cos t)(T - 20)$ where a is a constant to be found.

   (a) (3 pts) Is the equation linear? homogeneous? separable?

   (b) (10 pts) **Use the integrating factor method** to find the explicit function giving the object's temperature as a function of time.

10. [2360/050725 (15 pts)] Consider the system $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}}$ where $\mathbf{A} = \begin{bmatrix} a & 2 \\ 0 & 1 \end{bmatrix}$. Find all real values of a, if any, such that the system possesses the following stability and/or geometry properties. Recall that fixed points, critical points, equilibrium points and equilibrium solutions all refer to the same thing, that is, these are vectors which make $\vec{\mathbf{x}}' = \vec{\mathbf{0}}$. No partial credit available.

    (a) There exist nonisolated fixed points.

    (b) The isolated fixed point at $(0,0)$ is a center.

    (c) Any fixed point(s) is(are) stable.

    (d) The isolated fixed point at $(0,0)$ is a saddle.

    (e) The isolated fixed point at $(0,0)$ is an unstable degenerate or star node.

Short table of Laplace Transforms: $\mathscr{L}\{f(t)\} = F(s) \equiv \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$

In this table, a, b, c are real numbers with $c \geq 0$, and $n = 0, 1, 2, 3, \ldots$

$$\mathscr{L}\left\{t^n e^{at}\right\} = \frac{n!}{(s-a)^{n+1}} \qquad \mathscr{L}\left\{e^{at}\cos bt\right\} = \frac{s-a}{(s-a)^2+b^2} \qquad \mathscr{L}\left\{e^{at}\sin bt\right\} = \frac{b}{(s-a)^2+b^2}$$

$$\mathscr{L}\{\cosh bt\} = \frac{s}{s^2-b^2} \qquad \mathscr{L}\{\sinh bt\} = \frac{b}{s^2-b^2}$$

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{\mathrm{d}^n F(s)}{\mathrm{d}s^n} \qquad \mathscr{L}\left\{e^{at} f(t)\right\} = F(s-a) \qquad \mathscr{L}\{\delta(t-c)\} = e^{-cs}$$

$$\mathscr{L}\{t f'(t)\} = -F(s) - s\frac{\mathrm{d}F(s)}{\mathrm{d}s} \qquad \mathscr{L}\{f(t-c)\,\mathrm{step}(t-c)\} = e^{-cs}F(s) \qquad \mathscr{L}\{f(t)\,\mathrm{step}(t-c)\} = e^{-cs}\mathscr{L}\{f(t+c)\}$$

$$\mathscr{L}\left\{f^{(n)}(t)\right\} = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \cdots - f^{(n-1)}(0)$$