1. (16 pt) Determine whether the series is convergent or divergent.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{10}{\sqrt[3]{n^2+1000}}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{3}{(\arctan n)^3}$

Solution:

(a) Use the Limit Comparison Test and compare to the divergent p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$.

$$\begin{aligned}\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{10}{\sqrt[3]{n^2+1000}} \cdot \frac{n^{2/3}}{1} \\ &= \lim_{n\to\infty} 10 \cdot \sqrt[3]{\frac{n^2}{n^2+1000}} = 10 \cdot 1 = 10 > 0\end{aligned}$$

Therefore the given series also is $\boxed{\text{divergent}}$.

(b) By the Test for Divergence,

$$\lim_{n\to\infty} \frac{3}{(\arctan n)^3} = \lim_{n\to\infty} \frac{3}{(\pi/2)^3} \neq 0.$$

Therefore the series is $\boxed{\text{divergent}}$.

2. (22 pt)

(a) Determine whether the series $\displaystyle\sum_{n=3}^{\infty} (-1)^n \frac{\ln n}{n}$ is absolutely convergent, conditionally convergent, or divergent.

(b) Determine the interval of convergence, including any endpoints, for the series $\displaystyle\sum_{n=3}^{\infty} \frac{(\ln n)(x-1)^n}{n}$.

Solution:

(a) First check for absolute convergence. The series $\displaystyle\sum_{n=3}^{\infty} \frac{\ln n}{n}$ diverges by the Direct Comparison Test because

$$\frac{\ln n}{n} > \frac{1}{n} > 0 \quad \text{for } n \geq 3$$

and $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n}$ is a divergent p-series.

Next check for convergence using the Alternating Series Test with $b_n = \dfrac{\ln n}{n}$.

- The derivative $\dfrac{d}{dx}\left(\dfrac{\ln x}{x}\right) = \dfrac{1-\ln x}{x^2} < 0$ for $x \geq 3$, so b_n is decreasing.
- $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} \dfrac{\ln n}{n} \overset{LH}{=} \lim\limits_{n\to\infty} \dfrac{1/n}{1} = 0.$

Therefore $\sum\limits_{n=3}^{\infty}(-1)^n \dfrac{\ln n}{n}$ is convergent but not absolutely, so it is $\boxed{\text{conditionally convergent}}$.

(b) Apply the Ratio Test.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(\ln(n+1))(x-1)^{n+1}}{n+1}\cdot\frac{n}{(\ln n)(x-1)^n}\right| \\
&= \lim_{n\to\infty}\left|\frac{\ln(n+1)}{\ln n}\cdot\frac{n}{n+1}\cdot\frac{(x-1)^{n+1}}{(x-1)^n}\right| \\
&\overset{LH}{=} \lim_{n\to\infty}\frac{1/(n+1)}{1/n}\cdot\frac{n}{n+1}\cdot|x-1| \\
&= \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^2|x-1| \\
&\overset{LH}{=} |x-1|
\end{aligned}$$

The power series is absolutely convergent for $|x-1| < 1 \implies -1 < x-1 < 1 \implies 0 < x < 2$. Using the results from part (a), at the endpoint $x = 0$, the series $\sum\limits_{n=3}^{\infty}(-1)^n\dfrac{\ln n}{n}$ is convergent, and at the endpoint $x = 2$, the series $\sum\limits_{n=3}^{\infty}\dfrac{\ln n}{n}$ is divergent. Therefore the interval of convergence for the power series is $\boxed{[0,2)}$.

3. (18 pt) Let $g(x) = \dfrac{1}{(1-8x)^2}$.

(a) Find a power series representation for $g(x)$. Write your answer in sigma notation.

(b) Use your answer for part (a) to find a power series representation for $\displaystyle\int x^8 g(x)\,dx$.

Solution:

(a) The Maclaurin series for $\dfrac{1}{1-x}$ is $\sum\limits_{n=0}^{\infty} x^n$, so the series for $\dfrac{1}{1-8x}$ is $\sum\limits_{n=0}^{\infty} 8^n x^n$.

$$\begin{aligned}
g(x) = \frac{1}{(1-8x)^2} &= \frac{d}{dx}\left(\frac{1}{8}\cdot\frac{1}{1-8x}\right) \\
&= \frac{d}{dx}\left(\frac{1}{8}\sum_{n=0}^{\infty}8^n x^n\right) \\
&= \frac{1}{8}\sum_{n=0}^{\infty}8^n n x^{n-1}
\end{aligned}$$

$$= \boxed{\sum_{n=0}^{\infty} 8^{n-1} n x^{n-1}} = \boxed{\sum_{n=1}^{\infty} 8^{n-1} n x^{n-1}}$$

(b)

$$\begin{aligned}
\int x^8 g(x)\,dx &= \int x^8 \sum_{n=1}^{\infty} 8^{n-1} n x^{n-1}\,dx \\
&= \int \sum_{n=1}^{\infty} 8^{n-1} n x^{n+7}\,dx \\
&= \boxed{C + \sum_{n=1}^{\infty} \frac{8^{n-1} n}{n+8} x^{n+8}\,dx}
\end{aligned}$$

4. (18 pt) Let $f(x) = \sin(3x)$. Be sure to simplify your answers to the following problems.

   (a) Find $T_3(x)$, the 3rd degree Taylor polynomial of f, centered at 0, and use it to approximate the value of $\sin(1)$.

   (b) Use Taylor's Formula to find an error bound for the approximation.

Solution:

(a) The Maclaurin series for $\sin x$ is $\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$, so a series for $\sin(3x)$ is $\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{3^{2n+1}x^{2n+1}}{(2n+1)!}$.

Then $T_3(x) = 3x - \dfrac{3^3}{3!}x^3 = \boxed{3x - \dfrac{9}{2}x^3}$ and

$$\sin(1) \approx T_3\left(\frac{1}{3}\right) = 3 \cdot \frac{1}{3} - \frac{9}{2} \cdot \frac{1}{3^3} = 1 - \frac{1}{6} = \boxed{\frac{5}{6}}.$$

(b) The approximation error is

$$R_3(x) = \frac{f^{(4)}(z)}{4!}x^4 \quad \text{for } x = \frac{1}{3} \text{ and } 0 < z < 1/3.$$

The first four derivatives of $f(x)$ are

$$f'(x) = 3\cos(3x),\ f''(x) = -9\sin(3x),\ f'''(x) = -27\cos(3x), \text{ and } f^{(4)}(x) = 81\sin(3x).$$

Because $0 < z < 1/3$ implies $0 < \sin(3z) < \sin 1$,

$$\left|f^{(4)}(z)\right| = |81\sin(3z)| < 81\sin 1.$$

Therefore an error bound for the approximation is

$$|R_3(1/3)| < \frac{81\sin 1}{4!}\left(\frac{1}{3}\right)^4 = \frac{\sin 1}{4!} = \boxed{\frac{\sin 1}{24}}.$$

An error bound of $\boxed{1/24}$ also is acceptable because $\sin(3z) < 1$.

5. The following three problems are not related.

   (a) (8 pt) Use the binomial series representation for $(1+4x)^{3/4}$ to find the coefficient of the x^3 term. Fully simplify your answer.

   (b) (8 pt) Find the sum of the series

$$\frac{1}{e} - \frac{1}{3e^3} + \frac{1}{5e^5} - \frac{1}{7e^7} + \cdots.$$

   (c) (10 pt) Consider the parametric curve defined by $x = 3\tan^2 t,\ y = 3\sec^2 t$ for $0 \le t \le \pi/4$.

      i. Find the x and y coordinates of the initial and terminal points of the curve.

      ii. Eliminate the parameter to find a Cartesian equation for the curve.

Solution:

(a) $(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n \implies (1+4x)^{3/4} = \sum_{n=0}^{\infty} \binom{3/4}{n} 4^n x^n$.

Let $n = 3$. Then the coefficient of the x^3 term is $\binom{3/4}{3} 4^3 = \frac{\frac{3}{4}(-\frac{1}{4})(-\frac{5}{4})}{3!} \cdot 4^3 = \boxed{\frac{5}{2}}$.

(b) The Maclaurin series for $\tan^{-1} x$ is $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$, $R = 1$. Let $x = 1/e$.

Then $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left(\frac{1}{e}\right)^{2n+1} = \frac{1}{e} - \frac{1}{3e^3} + \frac{1}{5e^5} - \frac{1}{7e^7} + \cdots = \boxed{\tan^{-1}(1/e)}$.

(c) i. At the initial point where $t = 0$, the coordinates are $x = 3\tan^2(0) = \boxed{0}$, and $y = 3\sec^2(0) = \boxed{3}$.

At the terminal point where $t = \frac{\pi}{4}$, the coordinates are $x = 3\tan^2(\pi/4) = \boxed{3}$, and $y = 3\sec^2(\pi/4) = 3\left(\sqrt{2}\right)^2 = \boxed{6}$.

ii. Use a trig identity to find a Cartesian equation.

$$\sec^2 t = \tan^2 t + 1$$
$$3\sec^2 t = 3\tan^2 t + 3$$
$$\boxed{y = x + 3}$$