1. (11 points) Consider the vector field $\mathbf{F}(x, y, z) = \langle yz, -y^2z, yz^2\rangle$.
   (a) (4 points) Find the divergence of the vector field $\mathbf{F}$.
   (b) (4 points) Find the curl of the vector field $\mathbf{F}$.
   (c) (3 points) Is $\mathbf{F}$ incompressible? Briefly explain why or why not.

   Solution:

   (a) div $\mathbf{F} = \nabla \cdot \mathbf{F} = 0$.
   (b) curl $\mathbf{F} = \nabla \times \mathbf{F} = \langle z^2 + y^2, y, -z\rangle$.
   (c) Yes, $\mathbf{F}$ is incompressible since div $\mathbf{F} = \nabla \cdot \mathbf{F} = 0$.

2. (14 points) Evaluate $\displaystyle\int_0^2 \int_y^2 e^{x^2}\,dx\,dy$. (Hint: You may find it helpful to sketch the region and switch the order of integration.)

   Solution:

   If we switch the order of integration, we have

$$\begin{aligned}\int_0^2 \int_y^2 e^{x^2}\,dx\,dy &= \int_0^2 \int_0^x e^{x^2}\,dy\,dx \\ &= \int_0^2 xe^{x^2}\,dx.\end{aligned}$$

   We then apply the substitution $u = x^2$. So, $\frac{1}{2}du = x\,dx$ and the new limits of integration are $u = 0$ (lower) and $u = 4$ (upper):

$$\begin{aligned}\int_0^2 xe^{x^2}\,dx &= \frac{1}{2}\int_0^4 e^u\,du \\ &= \frac{e^4 - 1}{2}.\end{aligned}$$

3. (19 points) Captain Bonaventura Cavalieri is a pirate who likes to steal acorns from unsuspecting squirrels. He recently stole an acorn from Sam the Squirrel after Sam accidentally dropped it while running in a park. Captain Bonaventura Cavalieri stores these acorns in a vault built on the region of land on the xy-plane, $\mathcal{R}$, bounded by $x^2 + y^2 = 25$ where $x \leq 0$. The mass density of acorns in this vault is given by $\rho(x, y) = y^2$ kilograms per square meter.

   (a) (14 points) Evaluate an integral to determine the mass of acorns in this vault. (Include the correct units in your final answer.)
   (b) (5 points) Find the average value of $\rho(x, y)$ over the region $\mathcal{R}$. (Include the correct units in your final answer.)

   Solution:

(a)

$$\begin{aligned}
\iint_{\mathcal{R}} y^2\,dA &= \int_{\pi/2}^{3\pi/2}\int_0^5 (r\sin\theta)^2 r\,dr\,d\theta \\
&= \left(\int_{\pi/2}^{3\pi/2}\sin^2\theta\,d\theta\right)\left(\int_0^5 r^3\,dr\right) \\
&= \left(\frac{1}{2}\int_{\pi/2}^{3\pi/2} 1-\cos(2\theta)\,d\theta\right)\left(\frac{625}{4}\right) \\
&= \frac{625}{8}\left[\theta - \frac{1}{2}\sin(2\theta)\right]_{\pi/2}^{3\pi/2} \\
&= \frac{625\pi}{8} \text{ kilograms.}
\end{aligned}$$

(b) The area of the region is $\pi\cdot 5^2/2 = 25\pi/2$. So, the average value of $\rho(x,y)$ over $\mathcal{R}$ is

$$\frac{625\pi/8}{25\pi/2} = \frac{25}{4} \text{ kilograms per square meter.}$$

4. (17 points) Evaluate the surface integral $\iint_S x\,dS$ where S is the surface $z = x^2+y$ for $0\le x\le 1$ and $0\le y\le 3$.

Solution: We have $g(x,y,z) = x^2+y-z$. So, $\nabla g = \langle 2x, 1, -1\rangle$ which means $||\nabla g|| = \sqrt{2+4x^2}$. If we use $\mathbf{p} = \mathbf{k}$, then $|\nabla g\cdot\mathbf{p}| = 1$. This yields the integral

$$\begin{aligned}
\iint_S x\,dS &= \int_0^1\int_0^3 x\frac{||\nabla g||}{|\nabla g\cdot\mathbf{p}|}\,dy\,dx \\
&= \int_0^1\int_0^3 x\sqrt{2+4x^2}\,dy\,dx \\
&= 3\int_0^1 x\sqrt{2+4x^2}\,dx.
\end{aligned}$$

If we apply the substitution $u = 2+4x^2$, then $\frac{1}{8}\,du = x\,dx$, with new limits of integration $u = 2$ (lower) and $u = 6$ (upper):

$$\begin{aligned}
3\int_0^1 x\sqrt{2+4x^2}\,dx &= \frac{3}{8}\int_2^6 u^{1/2}\,du \\
&= \left[\frac{1}{4}u^{3/2}\right]_2^6 \\
&= \frac{6^{3/2}-2^{3/2}}{4}.
\end{aligned}$$

5. (18 points) Consider the polygonal region in the xy-plane with vertices $(1/2,1/2)$, $(1,1)$, $(1,0)$, and $(2,0)$. We will evaluate $\iint_{\mathcal{R}} 8xy\,dA$ by applying the change of variables $u = x+y$ and $v = x-y$. Let us proceed in the following steps:

(a) (4 points) Find the transformations of the vertices into (u,v) coordinates using the change of variables $u = x+y$ and $v = x-y$.

(b) (6 points) Sketch the polygonal region $\mathcal{S}$ in the uv-plane formed by the vertices you found in (a). Axes and the coordinates of vertices should be clearly labeled. Shade in the region itself.

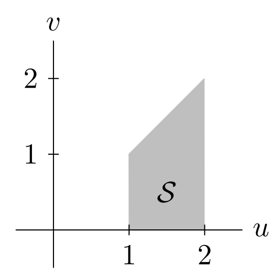
(c) (8 points) Determine the value of $\iint_{\mathcal{R}} 8xy\,dA$ by evaluating the appropriate integral over $\mathcal{S}$.

Solution:

(a) The vertices are noted in the left column below. We can use the given change of variable to generate the new vertices in the uv-plane:

(x,y)	(u,v)
$\left(\frac{1}{2},\frac{1}{2}\right)$	$(1,0)$
$(1,1)$	$(2,0)$
$(1,0)$	$(1,1)$
$(2,0)$	$(2,2)$

(b) The region $\mathcal{S}$ is plotted below.



(c) We need to determine the determinant of the Jacobian. To do that, we need to know $x = x(u,v)$ and $y = y(u,v)$. We see that

$$u + v = (x+y) + (x-y) = 2x \implies x = \frac{u+v}{2}$$

and

$$u - v = (x+y) - (x-y) = 2y \implies y = \frac{u-v}{2}.$$

So, the Jacobian is $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$ which has determinant $-1/2$. So,

$$\begin{aligned}
\iint_{\mathcal{R}} 8xy\,dA &= \iint_{\mathcal{S}} 8\left(\frac{u+v}{2}\right)\left(\frac{u-v}{2}\right)\left|\frac{-1}{2}\right|\,dv\,du \\
&= \int_1^2 \int_0^u u^2 - v^2\,dv\,du \\
&= \int_1^2 \left[u^2 v - (1/3)v^3\right]_0^u\,du \\
&= \frac{2}{3}\int_1^2 u^3\,du \\
&= \left[\frac{1}{6}u^4\right]_1^2 \\
&= \frac{5}{2}.
\end{aligned}$$

6. (21 points) The integral $V = \int_0^{2\pi} \int_2^{2\sqrt{3}} \int_2^{\sqrt{16-r^2}} r\,dz\,dr\,d\theta$ provides the volume of a solid region.

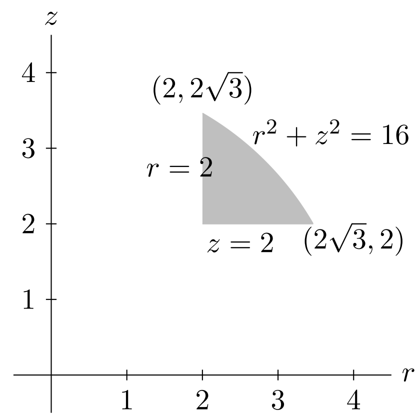
(a) (5 points) Make a clear sketch of the cross-section of the solid region in the rz-plane. Axes, intercepts, and curves should be clearly labeled. Shade in the region itself.

(b) (8 points) Express V as the sum of integral(s) in spherical coordinates using the order $d\phi\,d\rho\,d\theta$. Do **NOT** evaluate this integral.

(c) (8 points) Express V as the sum of integral(s) in Cartesian coordinates using the order $dz\,dx\,dy$. Do **NOT** evaluate this integral.

Solution:

(a) Here is a sketch of the region:



(b) We note that the ϕ will need to go from $r = 2$ to $z = 2$, so both of these curves must be converted to spherical coordinates, and solved for ϕ. So, we have

$$\begin{aligned} r &= 2 \\ \rho \sin \phi &= 2 \\ \phi &= \arcsin(2/\rho) \end{aligned}$$

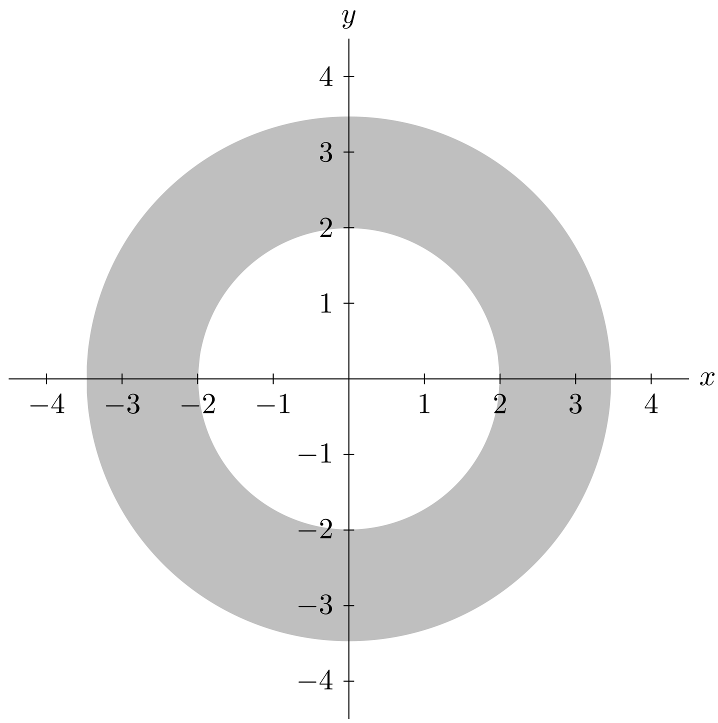
and

$$\begin{aligned} z &= 2 \\ \rho \cos \phi &= 2 \\ \phi &= \arccos(2/\rho). \end{aligned}$$

(One may also obtain $\phi = \arccos \sqrt{1 - 4/\rho^2}$ instead of $\phi = \arcsin(2/\rho)$.)

So, we have $V = \int_0^{2\pi} \int_{2\sqrt{2}}^{4} \int_{\arcsin(2/\rho)}^{\arccos(2/\rho)} \rho^2 \sin\phi\,d\phi\,d\rho\,d\theta$.

(c) It is helpful to consider the projection of this surface onto the xy-plane, which is the annulus with inner radius 2 and outer radius $2\sqrt{3}$:



We will need to integrate z from $z = 2$ to $z = \sqrt{16 - x^2 - y^2}$. After that, we will set up our integral by (1) integrating over the first quadrant and multiplying by 4 and (2) integrating over the larger disk and subtracting the integral for the smaller disk:

$$V = 4\left[\int_0^{2\sqrt{3}} \int_0^{\sqrt{12-y^2}} \int_2^{\sqrt{16-x^2-y^2}} \, dz \, dx \, dy - \int_0^2 \int_0^{\sqrt{4-y^2}} \int_2^{\sqrt{16-x^2-y^2}} \, dz \, dx \, dy\right].$$

An alternate solution is given by

$$V = 4\left[\int_2^{2\sqrt{3}} \int_0^{\sqrt{12-y^2}} \int_2^{\sqrt{16-x^2-y^2}} \, dz \, dx \, dy + \int_0^2 \int_{\sqrt{4-y^2}}^{\sqrt{12-y^2}} \int_2^{\sqrt{16-x^2-y^2}} \, dz \, dx \, dy\right].$$