1. [2360/041625 (10 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given. Please write your answers in a single column separate from any work you do to arrive at the answer.

   (a) The set $\left\{t^2, e^t\right\}$ forms a basis for the solution space of the differential equation $(t-1)y'' - ty' + y = 0$.

   (b) The potential energy of the conservative system governed by $2y'' + e^{-y} + 2y = 0$ is $y^2 - e^{-y}$.

   (c) The general solution of $2y'' + 24y = \sin\left(2\sqrt{3}t\right) + \cos t$ is bounded.

   (d) The equation $x'' + 2x' + x^2 = \sin t$ can be written as $\vec{\mathbf{u}}' = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}\vec{\mathbf{u}} + \begin{bmatrix} 0 \\ \sin t \end{bmatrix}$, where $\vec{\mathbf{u}} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$.

   (e) The correct form of the particular solution when using the method of undetermined coefficients to solve the equation $y^{(4)} - 4y''' + 13y'' = t^2 + e^{2t}\cos 3t + \sin 2t$ is $y_p = At^4 + Bt^3 + Ct^2 + te^{2t}\left(D\cos 3t + E\sin 3t\right) + F\cos 2t + G\sin 2t$.

SOLUTION:

(a) **FALSE** Although the Wronskian of the two functions, $W = t(t-2)e^t$, is not identically zero, showing that the functions are linearly independent, they do not form a basis of the solution space since t^2 is not a solution of the differential equation.

(b) **TRUE** The potential energy is $\displaystyle\int \left(e^{-y} + 2y\right) \mathrm{d}y = y^2 - e^{-y}$.

(c) **FALSE** A basis for the solution space of the associated homogeneous equation is $\left\{\cos 2\sqrt{3}t, \sin 2\sqrt{3}t\right\}$ meaning that the particular solution will contain either or both of $t\sin 2\sqrt{3}t$ and $t\cos 2\sqrt{3}t$. These grow without bound as $t \to \infty$ implying that the solutions are unbounded.

(d) **FALSE** The equation is nonlinear. With $u_1 = x$ and $u_2 = x'$ and $\vec{\mathbf{u}} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, we have

$$u_1' = x' = u_2$$
$$u_2' = x'' = \sin t - 2x' - x^2 = -u_1^2 - 2u_2 + \sin t$$
$$\vec{\mathbf{u}}' = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}' = \begin{bmatrix} u_2 \\ -u_1^2 - 2u_2 + \sin t \end{bmatrix}$$

(e) **TRUE** A basis for the solution space of the associated homogeneous problem is $\left\{1, t, e^{2t}\cos 3t, e^{2t}\sin 3t\right\}$.

2. [2360/041625 (20 pts)] Consider the linear operator $L(\vec{\mathbf{y}}) = ty'' - y' + \dfrac{y}{t}$. Assume $t > 0$.

   (a) (8 pts) Justify why the solution space, $\mathbb{S}$, of $L(\vec{\mathbf{y}}) = 0$ is $\mathbb{S} = \operatorname{span}\{t, t\ln t\}$.

   (b) (12 pts) Solve $L(\vec{\mathbf{y}}) = \ln t$.

SOLUTION:

(a)

$$t(t'') - t' + \frac{t}{t} = 0 - 1 + 1 = 0 \quad \checkmark$$

$$t(t\ln t)'' - (t\ln t)' + \frac{t\ln t}{t} = t\left(\frac{1}{t}\right) - (1 + \ln t) + \ln t = 1 - 1 - \ln t + \ln t = 0 \quad \checkmark$$

$$W[t, t\ln t](t) = \begin{vmatrix} t & t\ln t \\ 1 & 1 + \ln t \end{vmatrix} = t \not\equiv 0$$

Both functions are solutions and they are linearly independent. Since the solution space has dimension 2 (second order linear equation), the two functions form a basis of the solution space so that their span is the solution space.

(b) Getting the equation into the correct form, we need to solve $y'' - \frac{y'}{t} + \frac{y}{t^2} = \frac{\ln t}{t}$. We use Variation of Parameters with $y_1 = t, y_2 = t\ln t$ and $y_p = v_1(t)y_1 + v_2(t)y_2$.

$$v_1(t) = \int -\frac{t\ln t}{t}\left(\frac{\ln t}{t}\right)\,\mathrm{d}t = -\int \frac{(\ln t)^2}{t} \overset{u=\ln t}{=} -\frac{1}{3}(\ln t)^3$$

$$v_2(t) = \int \frac{t}{t}\left(\frac{\ln t}{t}\right)\,\mathrm{d}t = \int \frac{\ln t}{t}\,\mathrm{d}t \overset{u=\ln t}{=} \frac{1}{2}(\ln t)^2$$

$$y_p = -\frac{1}{3}(\ln t)^3(t) + \frac{1}{2}(\ln t)^2(t\ln t) = \frac{1}{6}t(\ln t)^3$$

$$y = y_h + y_p = c_1 t + c_2 t\ln t + \frac{1}{6}t(\ln t)^3 = t\left[c_1 + c_2\ln t + \frac{1}{6}(\ln t)^3\right]$$

■

3. [2360/041625 (20 pts)] Use the method of undetermined coefficients to solve the initial value problem

$$3y'' - 6y' - 9y = 12e^t\left(2e^{2t} + 3\right),\ y(0) = y'(0) = 0$$

SOLUTION:

The characteristic equation for the associated homogeneous problem is $3r^2 - 6r - 9 = 3(r^2 - 2r - 3) = 3(r-3)(r+1) = 0$ which has roots $r = -1, 3$, implying that the solution to the homogeneous problem is $y_h = c_1e^{3t} + c_2e^{-t}$. The forcing function is $24e^{3t} + 36e^t$ so that the guess for the particular solution is $y_p = Ate^{3t} + Be^t$. Substituting into the nonhomogeneous problem gives

$$3y_p'' - 6y_p' - 9y_p = 3\left(9Ate^{3t} + 6Ae^{3t} + Be^t\right) - 6\left(3Ate^{3t} + Ae^{3t} + Be^t\right) - 9\left(Ate^{3t} + Be^t\right)$$
$$= 12Ae^{3t} - 12Be^t = 24e^{3t} + 36e^t \implies A = 2, B = -3 \implies y_p = 2te^{3t} - 3e^t$$

We apply the initial conditions to $y(t) = y_h + y_p = c_1e^{3t} + c_2e^{-t} + 2te^{3t} - 3e^t$ and $y'(t) = 3c_1e^{3t} - c_2e^{-t} + 6te^{3t} + 2e^{3t} - 3e^t$ yielding.

$$y(0) = c_1 + c_2 - 3 = 0 \implies c_1 + c_2 = 3$$
$$y'(0) = 3c_1 - c_2 + 2 - 3 = 0 \implies 3c_1 - c_2 = 1$$

These give $4c_1 = 4 \implies c_1 = 1$ and $c_2 = 2$ so that the solution to the initial value problem is $y(t) = e^{3t} + 2e^{-t} + 2te^{3t} - 3e^t$. ■

4. [2360/041625 (40 pts)] The differential equation governing the motion of an harmonic oscillator can be written in the alternate form

$$\ddot{x} + 2a\dot{x} + \omega_0^2 x = g(t) \tag{1}$$

where $\omega_0 > 0$ is the circular frequency and $a \geq 0$. Use Eq. (1) to answer the following.

(a) (4 pts) Find a relationship between a and ω_0 that guarantees the oscillator passes through its equilibrium position at most once.

(b) (4 pts) If $g(t) = 10\cos 5t$ and the restoring (spring) constant is 100, find values of a and the mass of the object that put the oscillator into resonance.

(c) (32 pts) If $a = 4, \omega_0 = \sqrt{18}$ and $g(t) = 36$, answer the following.

i. (3 pts) Is the oscillator undamped, critically damped, overdamped or underdamped? Justify your answer.

ii. (25 pts) Use Laplace transforms to find the equation of motion assuming that the oscillator is initially at rest at its equilibrium position.

iii. (4 pts) Identify the steady state and transient portions of the solution, if they exist. If they don't exist, explain why not.

SOLUTION:

(a) The characteristic equation is $r^2 + 2ar + \omega_0^2 = 0$ having solutions $r = \frac{-2a \pm \sqrt{4a^2 - 4\omega_0^2}}{2} = -a \pm \sqrt{a^2 - \omega_0^2}$. In order for the oscillator to pass through its equilibrium position at most once, the roots of the characteristic equation must be real. Thus $a^2 - \omega_0^2 \geq 0 \implies a^2 \geq \omega_0^2 \implies |a| \geq |\omega_0| \implies a \geq \omega_0$ since both a and ω_0 are positive.

(b) The system must be undamped so that $a = 0$. Furthermore, the circular frequency of the oscillator must match that of the forcing function. Thus, we need $\omega_0 = 5 = \sqrt{\dfrac{100}{m}} \implies 25 = \dfrac{100}{m} \implies m = 4$.

(c) The initial value problem is $\ddot{x} + 8\dot{x} + 18x = 36,\ x(0) = \dot{x}(0) = 0$.

i. From part (a), $a^2 - \omega_0^2 = 16 - 18 < 0$ so the oscillator is underdamped. Note also that $8^2 - 4(1)(18) < 0$.

ii. The initial conditions are $x(0) = \dot{x}(0) = 0$.

$$\mathscr{L}\left\{\ddot{x} + 8\dot{x} + 18x = 36\right\}$$

$$s^2X(s) - sx(0) - \dot{x}(0) + 8[sX(s) - x(0)] + 18X(s) = \frac{36}{s}$$

$$\left(s^2 + 8s + 18\right)X(s) = \frac{36}{s}$$

$$X(s) = \frac{36}{s(s^2 + 8s + 18)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 8s + 18}$$

$$36 = A(s^2 + 8s + 18) + (Bs + C)s$$

$$s = 0 : 36 = A(18) \implies A = 2$$

$$s = 1 : 36 = 2(1 + 8 + 18) + B + C \implies B + C = -18$$

$$s = -1 : 36 = 2(1 - 8 + 18) + (-B + C)(-1) \implies B - C = 14$$

$$2B = -4 \implies B = -2 \text{ and } C = -16$$

$$X(s) = \frac{2}{s} - \frac{2s + 16}{s^2 + 8s + 18} = \frac{2}{s} - \frac{2s + 16}{(s + 4)^2 + 2} = \frac{2}{s} - 2\left[\frac{s + 4 + 4}{(s + 4)^2 + \left(\sqrt{2}\right)^2}\right]$$

$$= \frac{2}{s} - 2\left[\frac{s + 4}{(s + 4)^2 + \left(\sqrt{2}\right)^2} + \frac{4}{\sqrt{2}}\frac{\sqrt{2}}{(s + 4)^2 + \left(\sqrt{2}\right)^2}\right]$$

$$= \frac{2}{s} - 2\left[\frac{s + 4}{(s + 4)^2 + \left(\sqrt{2}\right)^2}\right] - 4\sqrt{2}\left[\frac{\sqrt{2}}{(s + 4)^2 + \left(\sqrt{2}\right)^2}\right]$$

$$x(t) = \mathscr{L}^{-1}\left\{X(s)\right\} = \mathscr{L}^{-1}\left\{\frac{2}{s} - 2\left[\frac{s + 4}{(s + 4)^2 + \left(\sqrt{2}\right)^2}\right] - 4\sqrt{2}\left[\frac{\sqrt{2}}{(s + 4)^2 + \left(\sqrt{2}\right)^2}\right]\right\}$$

$$= 2\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} - 2e^{-4t}\mathscr{L}^{-1}\left\{\frac{s}{s^2 + \left(\sqrt{2}\right)^2}\right\} - 4\sqrt{2}e^{-4t}\mathscr{L}^{-1}\left\{\frac{\sqrt{2}}{s^2 + \left(\sqrt{2}\right)^2}\right\}$$

$$= 2 - 2e^{-4t}\cos\sqrt{2}t - 4\sqrt{2}e^{-4t}\sin\sqrt{2}t = 2\left[1 - e^{-4t}\left(\cos\sqrt{2}t + 2\sqrt{2}\sin\sqrt{2}t\right)\right]$$

iii. The steady state solution is 2 and the transient solution is $-2e^{-4t}\left(\cos\sqrt{2}t + 2\sqrt{2}\sin\sqrt{2}t\right)$.

■

5. [2360/041625 (10 pts)] Consider the initial value problem $ty'' + t^2y' + 3y = tg(t),\ y(0) = -5, y'(0) = 0$ and suppose $\mathscr{L}\left\{g(t)\right\} = \dfrac{2s}{s^4 + 4}$. Using Laplace transforms find, **but do not solve**, the ODE that $Y(s) = \mathscr{L}\left\{y(t)\right\}$ must satisfy. Write your answer in the form $L(\overrightarrow{\mathbf{Y}}) = f(s)$.

<u>SOLUTION:</u>

Begin by noting that $\mathscr{L}\left\{tg(t)\right\} = (-1)^1 \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{2s}{s^4+4}\right) = -\left[\frac{(s^4+4)(2) - 2s(4s^3)}{(s^4+4)^2}\right] = \frac{2\left(3s^4-4\right)}{\left(s^4+4\right)^2}$.

$$\mathscr{L}\left\{ty'' + t^2y' + 3y\right\} = \mathscr{L}\left\{ty''\right\} + \mathscr{L}\left\{t^2y'\right\} + \mathscr{L}\left\{3y\right\} = \mathscr{L}\left\{tg(t)\right\}$$

$$(-1)^1\frac{\mathrm{d}}{\mathrm{d}s}\mathscr{L}\left\{y''\right\} + (-1)^2\frac{\mathrm{d}^2}{\mathrm{d}s^2}\mathscr{L}\left\{y'\right\} + 3\mathscr{L}\left\{y\right\} = \mathscr{L}\left\{tg(t)\right\}$$

$$-\frac{\mathrm{d}}{\mathrm{d}s}\left[s^2Y(s) - sy(0) - y'(0)\right] + \frac{\mathrm{d}^2}{\mathrm{d}s^2}\left[sY(s) - y(0)\right] + 3Y(s) = \frac{2\left(3s^4-4\right)}{\left(s^4+4\right)^2}$$

$$-\left[s^2Y'(s) + 2sY(s) - y(0)\right] + \frac{\mathrm{d}}{\mathrm{d}s}\left[sY'(s) + Y(s)\right] + 3Y(s) = \frac{2\left(3s^4-4\right)}{\left(s^4+4\right)^2}$$

$$-s^2Y'(s) - 2sY(s) - 5 + sY''(s) + Y'(s) + Y'(s) + 3Y(s) = \frac{2\left(3s^4-4\right)}{\left(s^4+4\right)^2}$$

$$sY''(s) + (2-s^2)Y'(s) + (3-2s)Y(s) = 5 + \frac{2\left(3s^4-4\right)}{\left(s^4+4\right)^2}$$

$\blacksquare$