- This exam is worth 100 points and has 5 problems.
- Show all work and simplify your answers! Answers with no justification will receive no points unless otherwise noted.
- Begin each problem on a new page.
- **DO NOT LEAVE THE EXAM UNTIL YOUR HAVE SATISFACTORILY SCANNED AND UPLOADED YOUR EXAM TO GRADESCOPE.**
- You are taking this exam in a proctored and honor code enforced environment. No calculators, cell phones, or other electronic devices or the internet are permitted during the exam. You are allowed one 8.5"× 11" crib sheet with writing on one side.

0. At the top of the first page that you will be scanning and uploading to Gradescope, write the following statement and sign your name to it: "I will abide by the CU Boulder Honor Code on this exam." **FAILURE TO INCLUDE THIS STATEMENT AND YOUR SIGNATURE MAY RESULT IN A PENALTY.**

1. [2360/041625 (10 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given. Please write your answers in a single column separate from any work you do to arrive at the answer.

   (a) The set $\left\{t^2, e^t\right\}$ forms a basis for the solution space of the differential equation $(t-1)y'' - ty' + y = 0$.

   (b) The potential energy of the conservative system governed by $2y'' + e^{-y} + 2y = 0$ is $y^2 - e^{-y}$.

   (c) The general solution of $2y'' + 24y = \sin\left(2\sqrt{3}t\right) + \cos t$ is bounded.

   (d) The equation $x'' + 2x' + x^2 = \sin t$ can be written as $\vec{\mathbf{u}}' = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \vec{\mathbf{u}} + \begin{bmatrix} 0 \\ \sin t \end{bmatrix}$, where $\vec{\mathbf{u}} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$.

   (e) The correct form of the particular solution when using the method of undetermined coefficients to solve the equation $y^{(4)} - 4y''' + 13y'' = t^2 + e^{2t}\cos 3t + \sin 2t$ is $y_p = At^4 + Bt^3 + Ct^2 + te^{2t}\left(D\cos 3t + E\sin 3t\right) + F\cos 2t + G\sin 2t$.

2. [2360/041625 (20 pts)] Consider the linear operator $L(\vec{\mathbf{y}}) = ty'' - y' + \dfrac{y}{t}$. Assume $t > 0$.

   (a) (8 pts) Justify why the solution space, $\mathbb{S}$, of $L(\vec{\mathbf{y}}) = 0$ is $\mathbb{S} = \text{span}\left\{t, t\ln t\right\}$.

   (b) (12 pts) Solve $L(\vec{\mathbf{y}}) = \ln t$.

3. [2360/041625 (20 pts)] Use the method of undetermined coefficients to solve the initial value problem

$$3y'' - 6y' - 9y = 12e^t\left(2e^{2t} + 3\right),\ y(0) = y'(0) = 0$$

MORE PROBLEMS AND LAPLACE TRANSFORM TABLE BELOW/ON REVERSE

4. [2360/041625 (40 pts)] The differential equation governing the motion of an harmonic oscillator can be written in the alternate form

$$\ddot{x} + 2a\dot{x} + \omega_0^2 x = g(t) \tag{1}$$

where $\omega_0 > 0$ is the circular frequency and $a \geq 0$. Use Eq. (1) to answer the following.

(a) (4 pts) Find a relationship between a and ω_0 that guarantees the oscillator passes through its equilibrium position at most once.

(b) (4 pts) If $g(t) = 10\cos 5t$ and the restoring (spring) constant is 100, find values of a and the mass of the object that put the oscillator into resonance.

(c) (32 pts) If $a = 4, \omega_0 = \sqrt{18}$ and $g(t) = 36$, answer the following.

i. (3 pts) Is the oscillator undamped, critically damped, overdamped or underdamped? Justify your answer.

ii. (25 pts) Use Laplace transforms to find the equation of motion assuming that the oscillator is initially at rest at its equilibrium position.

iii. (4 pts) Identify the steady state and transient portions of the solution, if they exist. If they don't exist, explain why not.

5. [2360/041625 (10 pts)] Consider the initial value problem $ty'' + t^2y' + 3y = tg(t),\ y(0) = -5, y'(0) = 0$ and suppose $\mathscr{L}\{g(t)\} = \dfrac{2s}{s^4+4}$. Using Laplace transforms find, **but do not solve**, the ODE that $Y(s) = \mathscr{L}\{y(t)\}$ must satisfy. Write your answer in the form $L(\vec{\mathbf{Y}}) = f(s)$.

Short table of Laplace Transforms: $\mathscr{L}\{f(t)\} = F(s) \equiv \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$

In this table, a, b, c are real numbers with $c \geq 0$, and $n = 0, 1, 2, 3, \ldots$

$$\mathscr{L}\left\{t^n e^{at}\right\} = \frac{n!}{(s-a)^{n+1}} \qquad \mathscr{L}\left\{e^{at}\cos bt\right\} = \frac{s-a}{(s-a)^2+b^2} \qquad \mathscr{L}\left\{e^{at}\sin bt\right\} = \frac{b}{(s-a)^2+b^2}$$

$$\mathscr{L}\{\cosh bt\} = \frac{s}{s^2-b^2} \qquad \mathscr{L}\{\sinh bt\} = \frac{b}{s^2-b^2}$$

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{\mathrm{d}^n F(s)}{\mathrm{d}s^n} \qquad \mathscr{L}\left\{e^{at} f(t)\right\} = F(s-a) \qquad \mathscr{L}\{\delta(t-c)\} = e^{-cs}$$

$$\mathscr{L}\{tf'(t)\} = -F(s) - s\frac{\mathrm{d}F(s)}{\mathrm{d}s} \qquad \mathscr{L}\{f(t-c)\,\mathrm{step}(t-c)\} = e^{-cs}F(s) \qquad \mathscr{L}\{f(t)\,\mathrm{step}(t-c)\} = e^{-cs}\mathscr{L}\{f(t+c)\}$$

$$\mathscr{L}\left\{f^{(n)}(t)\right\} = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \cdots - f^{(n-1)}(0)$$
