1. (32 pts) Evaluate each of the following.

   (a) $\displaystyle\int_0^1 \frac{2x-5}{\sqrt[3]{x^2-5x+6}}\,dx$

   (b) $\displaystyle\int (\tan\theta\cos\theta + \sin\theta)\,d\theta$

   (c) $\displaystyle\int_{-1}^{2} \left|2x^2-4x\right|\,dx$

   Solution:

   (a) Apply u-substitution:

$$u = x^2 - 5x + 6$$
$$du = (2x-5)dx$$

   Also,

$$x = 0 \quad \Rightarrow \quad u = 0^2 - (5)(0) + 6 = 6$$
$$x = 1 \quad \Rightarrow \quad u = 1^2 - (5)(1) + 6 = 2$$

   Therefore,

$$\begin{aligned}\int_0^1 \frac{2x-5}{\sqrt[3]{x^2-5x+6}}\,dx &= \int_6^2 u^{-1/3}\,du \\ &= \frac{3}{2}\,u^{2/3}\Big|_6^2 \\ &= \boxed{\frac{3}{2}\left(2^{2/3} - 6^{2/3}\right)}\end{aligned}$$

(b)

$$\int (\tan\theta\cos\theta + \sin\theta)\, d\theta = \int \left(\frac{\sin\theta}{\cos\theta}\cdot\cos\theta + \sin\theta\right) d\theta$$

$$= \int 2\sin\theta d\theta$$

$$= \boxed{-2\cos\theta + C}$$

(c) The curve $y = 2x^2 - 4x = 2x(x-2)$ is a concave-upwards parabola whose roots are $x = 0$ and $x = 2$. So, the quantity $2x^2 - 4x$ is positive on the intervals $(-\infty, 0) \cup (2, \infty)$ and it is negative on the interval $(0, 2)$. For the purposes of this particular problem, it is sufficient to know that the quantity $2x^2 - 4x$ is non-negative on the interval $[-1, 0]$ and it is non-positive on the interval $[0, 2]$.

$$\left|2x^2 - 4x\right| = \begin{cases} 2x^2 - 4x & , \ -1 \le x \le 0 \\ 4x - 2x^2 & , \ \ 0 < x \le 2 \end{cases}$$

$$\int_{-1}^{2} \left|2x^2 - 4x\right|\ dx = \int_{-1}^{0} \left(2x^2 - 4x\right)\ dx + \int_{0}^{2} \left(4x - 2x^2\right)\ dx$$

$$= \left(\frac{2x^3}{3} - 2x^2\right)\Bigg|_{-1}^{0} + \left(2x^2 - \frac{2x^3}{3}\right)\Bigg|_{0}^{2}$$

$$= 0 - \left(-\frac{2}{3} - 2\right) + \left(8 - \frac{16}{3}\right) - 0$$

$$= 10 - \frac{14}{3} = \boxed{\frac{16}{3}}$$

2. (20 pts) The following are unrelated.

   (a) Evaluate: $\displaystyle\sum_{i=1}^{n} \frac{1}{n}\left(\left(\frac{i}{n}\right)^3 + 2\right)$.

   (b) Evaluate the limit $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left(\left(\frac{i}{n}\right)^3 + 2\right)$ using summation formulas or by evaluating an appropriate definite integral. Recall that you may not use L'Hospital's rule or dominance of powers arguments.

Solution:

(a)

$$\begin{aligned}
\sum_{i=1}^{n} \frac{1}{n}\left(\left(\frac{i}{n}\right)^3 + 2\right) &= \frac{1}{n^4}\sum_{i=1}^{n} i^3 + \frac{2}{n}\cdot n \\
&= \frac{1}{n^4}\cdot\left[\frac{n(n+1)}{2}\right]^2 + 2 \\
&= \frac{(n+1)^2}{4n^2} + 2 \\
&= \frac{n^2+2n+1}{4n^2} + \frac{8n^2}{4n^2} \\
&= \boxed{\frac{9n^2+2n+1}{4n^2}}
\end{aligned}$$

(b) **Method 1:**

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\left(\left(\frac{i}{n}\right)^3+2\right) = \lim_{n\to\infty}\frac{9n^2+2n+1}{4n^2}$$

$$= \lim_{n\to\infty}\frac{9+2/n+1/n^2}{4} = \boxed{\frac{9}{4}}$$

Method 2 (one possible approach): We can express the limit as a definite integral of a function on interval $[0, 1]$. The interval $[0, 1]$ is divided into n equal subintervals, each having a width of $\Delta x = (b-a)/n = 1/n$ with $a = 0$ and $b = 1$. Then the location of the righthand boundary of each subinterval i is $x_i = a + i\Delta x = i/n$.

Suppose that a rectangle is constructed on each subinterval i such that the height of rectangle i is determined by the value of the function $f(x) = x^3 + 2$ evaluated at the righthand boundary x_i. The corresponding Riemann sum is:

$$R_n = \sum_{i=1}^{n}\Delta x \cdot f(x_i) = \sum_{i=1}^{n}\frac{1}{n}\left(\left(\frac{i}{n}\right)^3+2\right)$$

Therefore,

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\left(\left(\frac{i}{n}\right)^3+2\right) = \int_0^1 (x^3+2)dx$$

$$= \left(\frac{x^4}{4}+2x\right)\Big|_0^1$$

$$= \frac{1}{4}+2 = \boxed{\frac{9}{4}}$$

3. (12 pts) Clearly sketch the graph of a function $h(x)$ that satisfies the following properties (label any extrema, inflection point(s), and asymptote(s)):

(i) $h'(x) > 0$ if $-2 < x < 2$

(ii) $h'(x) < 0$ if $-\infty < x < -2$ or $2 < x < \infty$

(iii) $h'(-2) = 0$

(iv) $\lim_{x \to 2^-} h'(x) = 3$

(v) $\lim_{x \to 2^+} h'(x) = -3$

(vi) $h''(x) > 0$ if $x \neq 2$

Solution:

Property (i) indicates that h is increasing on the interval $(-2, 2)$.

Property (ii) indicates that h is decreasing on the intervals $(-\infty, -2) \cup (2, \infty)$.

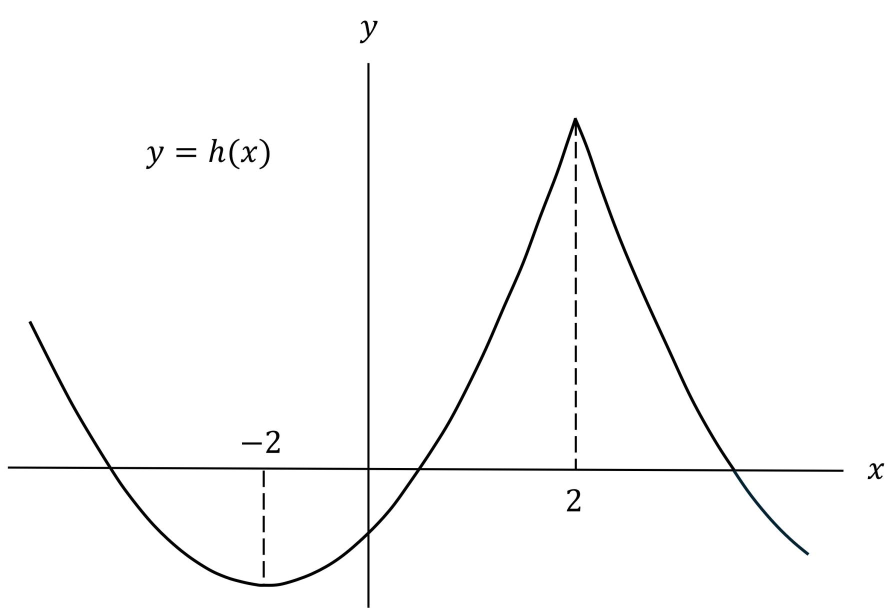
Property (iii) indicates that $y = h(x)$ has a horizontal tangent line at $x = -2$.

Property (iv) indicates that the slope of the tangent line to $y = h(x)$ approaches a value of 3 as x approaches a value of 2 from the left.

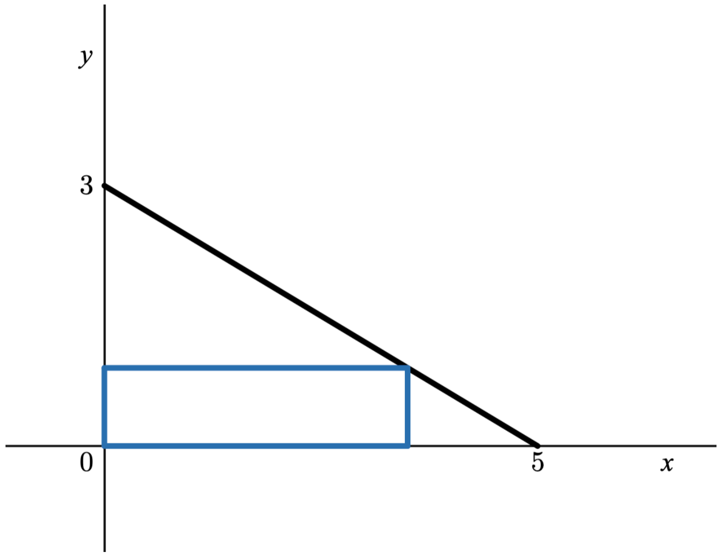
Property (v) indicates that the slope of the tangent line to $y = h(x)$ approaches a value of -3 as x approaches a value of 2 from the right.

Property (vi) indicates that $y = h(x)$ is concave up everywhere except at $x = 2$.

The following is an example of a function curve that satisfies all of the preceding criteria.



4. (12 pts) Suppose a rectangle has its left side lying on the y-axis, its bottom side lying on the x-axis, and the upper right corner touching the line that crosses through the points (0,3) and (5,0) (see diagram below). Find the dimensions of the rectangle that maximize the area of the rectangle. For full credit, verify that your final answer is a maximum value.



Solution:

The slope of the line that crosses through the points $(0, 3)$ and $(5, 0)$ is $\frac{\Delta y}{\Delta x} = \frac{0-3}{5-0} = -\frac{3}{5}$ and its y-intercept is 3.

So, the equation of the line can be written as $y = -\frac{3}{5}x + 3$.

For any value of x on the interval $(0, 5)$, the coordinates of the upper right corner of the rectangle are given by (x, y), where $y = -\frac{3}{5}x + 3$. Therefore, for any value of x on the interval $(0, 5)$, the area of the corresponding rectangle is:

$$A = (\text{width})(\text{height}) = xy = x\left(-\frac{3}{5}x + 3\right) = -\frac{3}{5}x^2 + 3x$$

The critical numbers of $A(x) = -\frac{3}{5}x^2 + 3x$ on its domain $(0, 5)$ are the values of x for which $A'(x) = 0$, so, $-\frac{6}{5}x + 3 = 0$ with solution $x = \frac{5}{2}$.

The only critical number of A on its domain is $x = \frac{5}{2}$, so this is the only possible location of a local extremum.

To verify that A does, in fact, attain a local maximum value at $x = \frac{5}{2}$, either the First Derivative Test or the Second Derivative Test can be applied.

Method 1: Second Derivative Test:

$$A'(x) = -\frac{6}{5}x + 3$$

$$A''(x) = -\frac{6}{5} < 0$$

Therefore, the Second Derivative Test indicates that A attains a local maximum value at $x = \frac{5}{2}$.

Method 2: First Derivative Test:

$$A'(x) = -\frac{6}{5}x + 3$$

A' transitions from positive to negative at $x = \frac{5}{2}$ so that the First Derivative Test confirms that A attains a local maximum value at $x = \frac{5}{2}$.

When the rectangle has a width of $x = \frac{5}{2}$, its corresponding height is $y = -\frac{3}{5}\left(\frac{5}{2}\right) + 3 = \frac{3}{2}$. Therefore, the dimensions of the rectangle with the greatest area are:

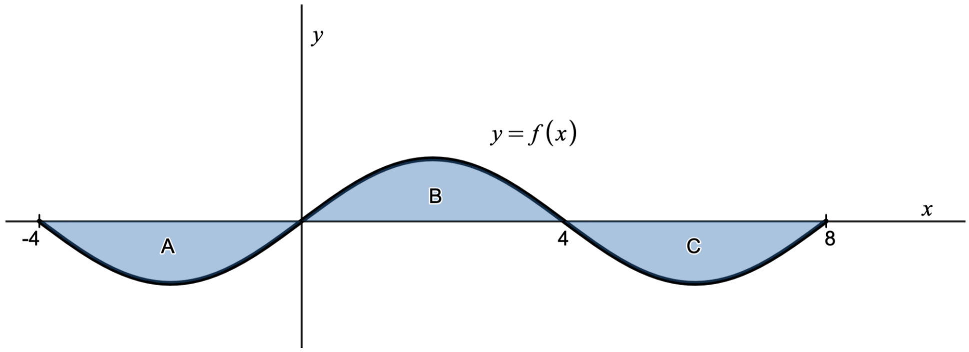
width $= x = \boxed{5/2}$

height $= y = \boxed{3/2}$

(Note that since A is continuous and has only one critical number on its domain, the local maximum of A at $x = \frac{5}{2}$ is also the absolute maximum of A on its domain.)

5. (24 pts) The following are unrelated:

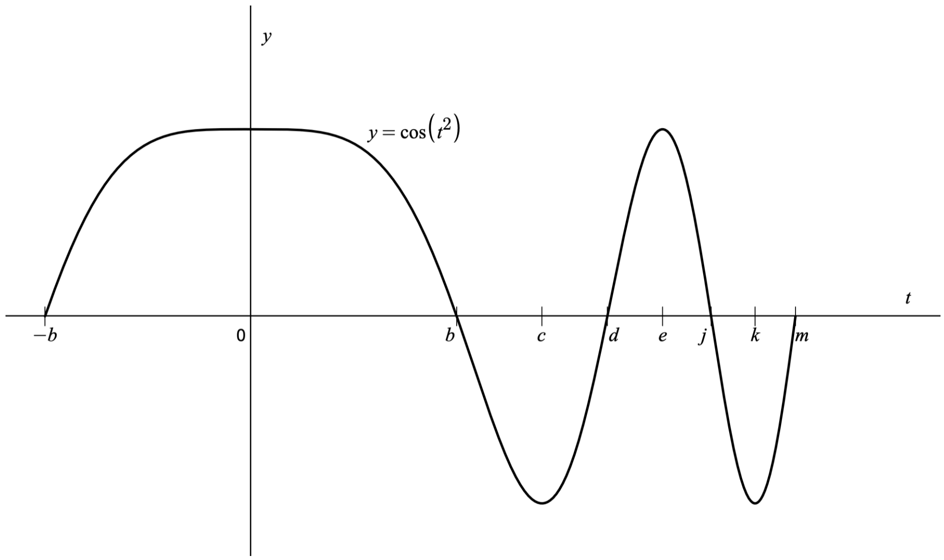
(a) Each of the regions, A, B, and C bounded by the graph of f and the x-axis, has an area of 5. Find the value of $\int_{-4}^{8} (f(x) + x + 5)\ dx$.



Solution:

$$\begin{aligned}
\int_{-4}^{8} (f(x) + x + 5)\ dx &= \int_{-4}^{8} f(x)\,dx + \int_{-4}^{8} x\,dx + \int_{-4}^{8} 5\,dx \\
&= (-5 + 5 - 5) + \frac{x^2}{2}\Bigg|_{-4}^{8} + 5x\Bigg|_{-4}^{8} \\
&= -5 + \left(\frac{64}{2} - \frac{16}{2}\right) + (40 - (-20)) \\
&= -5 + 24 + 60 = \boxed{79}
\end{aligned}$$

(b) Consider the function $y = \cos\left(t^2\right)$ shown below on domain $[-b, m]$. Let $g(x) = \int_{-b}^{x} \cos\left(t^2\right)\, dt$. Answer the following questions. Your answers to parts (iv) and (v) will be in terms of b, c, d, e, j, k, and m. **No justification is needed for this problem.**



i. What root (if any) of $y = \cos\left(t^2\right)$ would Newton's Method find if the initial guess was $t = c$?
ii. Find $g'(x)$
iii. Find $g''(x)$
iv. On which interval(s) is g decreasing?
v. At what x-value(s) does g have local minimum values?

Solution:

i. $\boxed{\text{None}}$ The function $y = \cos\left(t^2\right)$ has a horizontal tangent line at $t = c$, which does not intersect the t axis.

ii. According to part 1 of the Fundamental Theorem of Calculus,

$$g'(x) = \frac{d}{dx} \int_{-b}^{x} \cos\left(t^2\right) dt = \boxed{\cos\left(x^2\right)}$$

iii. $g''(x) = \dfrac{d}{dx}\left[\cos\left(x^2\right)\right] = -\sin\left(x^2\right) \cdot \dfrac{d}{dx}\left[x^2\right] = \boxed{-2x\sin\left(x^2\right)}$

iv. $\boxed{(b, d) \cup (j, m)}$ The function $g(x) = \int_{-b}^{x} \cos\left(t^2\right) dt$ represents the cumulative net area between the curve $y = \cos\left(t^2\right)$ and the t axis, from $t = -b$ to $t = x$. On the intervals $(b, d) \cup (j, m)$, the function $y = \cos\left(t^2\right)$ is negative, which leads to a decrease in the cumulative net area between the curve and the t axis, which corresponds to a decrease in $g(x)$.

v. $x = \boxed{d}$ As was stated in the preceding paragraph, the cumulative net area between the curve and the t axis decreases on the interval (b, d). Similarly, the cumulative area increases on the interval (d, f) since the function $y = \cos\left(t^2\right)$ is positive on that interval. Because the cumulative area transitions from decreasing to increasing at $t = d$, the function $g(x)$ has a local minimum value there. (Note that the function $g(x)$ does not have a local minimum at $t = m$ because functions can not have local extrema at the boundaries of their domains.)