APPM 3570/STAT 3100 Exam 2 Spring 2025

NAME: ______________________________

SECTION: (Circle One) 001 at 10:10 am *or* 002 at 2:30 pm

Instructions:

1. Calculators are permitted.
2. Notes, your text and other books, cell phones, and other electronic devices are not permitted—except for calculators or as needed to view and upload your work.
3. Justify your answers, show all work.
4. When you have completed the exam, go to the uploading area in the room and scan your exam and upload it to Gradescope.
5. Don't forget to scan any pages you used for extra space!
6. Verify that everything has been uploaded correctly and the pages have been associated to the correct problems.
7. Turn in your hardcopy exam.

On my honor as a University of Colorado Boulder student, I have neither given nor received unauthorized assistance on this work.

Signature: ______________________ Date: ____________

Duration: 90 minutes

Problem 1. (26 points.) Suppose Y is a random variable with pdf

$$f_Y(y) = \begin{cases} 3(1-y)^2 & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the expectation of Y.

(b) Find the variance of Y.

(c) Find the pdf of the random variable J when $J = 2Y - 1$.

Solution:

(a) (8 points.)
$\mathrm{E}[Y] = \int_0^1 3y(1-y)^2\,dy = \frac{1}{4}$

(b) (9 points.)
$\mathrm{E}[Y^2] = \int_0^1 3y^2(1-y)^2\,dy = \frac{1}{10}$

$$\mathrm{Var}(Y) = \frac{1}{10} - \frac{1}{16} = \frac{3}{80}$$

(c) (9 points) Since Y is continuous and $g(y)$ is monotonic and differentiable over the range of y, then $f_J(j) = 3(1 - \frac{j+1}{2})^2 \left(\frac{1}{2}\right)$ for $-1 < j < 1$

$$f_J(j) = \begin{cases} \frac{3}{8}(1-j)^2 & -1 < j < 1 \\ 0 & \text{otherwise} \end{cases}$$

(Use the back if additional space is needed!)

Problem 2. (21 points.) A book containing 150 pages has 100 misprints. Assume that the number of misprints in 150 pages follows a Poisson distribution. Recall that the Poisson distribution has the pmf:

$$P(X = i) = e^{-\lambda}\frac{\lambda^i}{i!} \quad \text{for } \lambda > 0$$

(a) Find the probability that a particular page contains no misprints.

(b) Find the probability that a particular page contains at least 2 misprints.

(c) Find the expected number of errors in any collection of 10 pages.

Solution:

(a) (7 points.) Since the number of errors in 150 pages is distributed as Poisson with $\lambda = 100$, then the number of errors on one page is distributed as Poisson with $\lambda = \frac{100}{150} = \frac{2}{3}$, then $P(X = 0) = e^{-\frac{2}{3}}\frac{\frac{2}{3}^0}{0!} = e^{-\frac{2}{3}} \approx .5134$

(b) (7 points.) $P(X \geq 2) = 1-(P(X = 0)+P(X = 1)) = 1-\left[e^{-\frac{2}{3}}\left(\frac{\frac{2}{3}^0}{0!} + \frac{\frac{2}{3}^1}{1!}\right)\right] \approx 1-.8557 = .1443$

(c) (7 points.) Since the number of errors in 150 pages is distributed as Poisson with $\lambda = 100$, then the number of errors on ten pages is distributed as Poisson with $\lambda = \frac{100}{150} \cdot 10 = \frac{20}{3}$, and the expected number of errors on ten pages is $\frac{20}{3}$.

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Problem 3. (24 points.)

(a) A certain variety of pine tree has a mean trunk diameter of $\mu = 150\,\text{cm}$ and a variance of $\sigma^2 = 900\,\text{cm}$. The diameter of the pine tree trunk in this area follows a normal distribution.

(i) What is the probability that a pine tree that is randomly selected from the area has a diameter between 100 and 180 centimeters?

(ii) If three pine trees are randomly selected, and each has a diameter measurement that is independent of the others, what is the distribution of the sum of their trunk diameters (the three trunk diameters added together)?

(b) (This problem is unrelated to any previous problem.) Let X be a random variable such that $\mathrm{P}(X = 1) = 1 - \mathrm{P}(X = 0) > 0$. If $3\mathrm{Var}(X) = 2\mathrm{E}[X]$, find $\mathrm{P}(X = 0)$.

Solution:

(a) (i) (8 points.) Let X denote the diameter of the selected pine tree. $\mathrm{P}(100 < X < 180) = \mathrm{P}(\frac{100-150}{30} < Z < \frac{180-150}{30}) = \mathrm{P}(-\frac{5}{3} < Z < 1) \approx \mathrm{P}(-1.67 < Z < 1) = \mathrm{P}(Z < 1) - \mathrm{P}(Z < -1.67) \approx .8413 - (1 - .9525) = .7938.$

(ii) (8 points.) Let X_i denote the diameter of the i-th selected pine tree, $i = 1, 2, 3$. $X_1 + X_2 + X_3 \sim \mathcal{N}(450,\ \sigma^2 = 2700)$.

(b) (8 points.) X is a Bernoulli random variable. Let $p = \mathrm{P}(X = 1)$ and thus $\mathrm{E}[X] = p$ and $\mathrm{Var}(X) = p(1 - p)$.

$$\begin{aligned}
3\mathrm{Var}(X) &= 2\mathrm{E}[X] \\
3(p)(1-p) &= 2p \\
3p - 3p^2 &= 2p \\
p &= 3p^2 \\
p &= \frac{1}{3}
\end{aligned}$$

Thus, $\mathrm{P}(X = 0) = 1 - \mathrm{P}(X = 1) = 1 - \frac{1}{3} = \frac{2}{3}$.

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Problem 4. (29 points.) Let X and Y be random variables with joint probability density function:

$$f_{X,Y}(x,y) = \begin{cases} 12xy(1-x) & 0 < x < 1,\ 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Are X and Y independent? Justify your answer.

(b) Find the marginal probability density function of X.

(c) Find $\mathrm{E}[X]$.

(d) Write the specific expression for the probability that $3Y > X$, complete with any integration symbols and bounds that are needed, but **do not solve**.

Solution:

(a) (7 points.) Yes, X and Y are independent. The joint pdf can be factored into a function that depends on x and a function that depends on y and the bounds of integration are rectangular.

(b) (7 points.)

$$\begin{aligned} f_X(x) &= \int_0^1 12xy(1-x)\,dy \\ &= \int_0^1 (12xy - 12x^2y)\,dy \\ &= (6xy^2 - 6x^2y^2)\Big|_0^1 \\ &= 6x(1-x) \quad 0 < x < 1 \end{aligned}$$

$$f_X(x) = \begin{cases} 6x(1-x) & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

(c) (7 points.)

$$\begin{aligned} \mathrm{E}[X] &= \int_0^1 6x^2(1-x)\,dx \\ &= \int_0^1 (6x^2 - 6x^3)\,dx \\ &= (2x^3 - \frac{3}{2}x^4)\Big|_0^1 \\ &= 2 - \frac{3}{2} \\ &= \frac{1}{2} \end{aligned}$$

(d) (8 points.)

$$\mathrm{P}(3Y > X) = \mathrm{P}\left(Y > \frac{X}{3}\right) = \int_0^1 \int_{\frac{x}{3}}^1 12xy(1-x)\,dy\,dx$$

Equivalently,

$$\mathrm{P}(3Y > X) = \mathrm{P}\left(Y > \frac{X}{3}\right) = \int_{\frac{1}{3}}^1 \int_0^1 12xy(1-x)\,dx\,dy + \int_0^{\frac{1}{3}} \int_0^1 12xy(1-x)\,dx\,dy$$

(Use the back if additional space is needed!)

Standard Normal Table

$\Phi(z) = P(Z \leq z)$ for $Z \sim N(0,1)$

	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990