1. (17 pts) A population of animals is increasing according to an exponential growth model. The initial population is 100 animals, and the population at $t = 10$ years is 300 animals.

   (a) What was the animal population at $t = 5$ years? For full credit, express your answer without using any logarithmic or exponential terms.

   Solution:

   Let $P(t)$ represent the population size at time t years. Since the population is increasing according to an exponential growth model, we have $P(t) = P_0 e^{kt}$, where $P_0 = P(0)$ and k is the relative growth rate.

   $P_0 = P(0) = 100$, so that $P(t) = 100e^{kt}$.

$$\begin{aligned} P(10) = 100e^{10k} &= 300 \\ e^{10k} &= 3 \\ \ln\left(e^{10k}\right) &= \ln 3 \\ 10k &= \ln 3 \\ k &= \frac{\ln 3}{10} \end{aligned}$$

   So, the exponential growth model for this population is $P(t) = 100\, e^{(\ln 3/10)\, t}$.

$$\begin{aligned} P(5) &= 100\, e^{(\ln 3/10)\cdot 5} \\ &= 100\, e^{0.5 \ln 3} \\ &= 100\, e^{\ln(3^{0.5})} \\ &= 100 \cdot 3^{0.5} \\ &= \boxed{100\sqrt{3}} \text{ animals} \end{aligned}$$

(b) How long will it take the population to increase from its initial size to a population of 2500 animals? Provide an exact expression for your answer and include the correct unit of measurement.

Solution:

$$\begin{aligned} 100\, e^{(\ln 3/10)\, t} &= 2500 \\ e^{(\ln 3/10)\, t} &= 25 \\ \ln\left[e^{(\ln 3/10)\, t}\right] &= \ln(25) \\ (\ln 3/10)\, t &= \ln(25) \\ t &= \boxed{\frac{10\ln(25)}{\ln 3} \text{ years}} \end{aligned}$$

(c) What was the instantaneous population growth rate (animals per year) at $t = 10$? Simplify your answer.

Solution:

The instantaneous population growth rate is given by $P^{'}(t) = \dfrac{dP}{dt} = kP(t)$.

Therefore, $P^{'}(10) = \left(\dfrac{\ln 3}{10}\right) \cdot P(10) = \left(\dfrac{\ln 3}{10}\right)(300) = \boxed{30\ln 3}$ animals per year.

2. (22 pts) Parts (a) and (b) are not related.

   (a) Consider the function $f(x) = \dfrac{2^x}{5 + 2^x}$.

      i. Explain why f is invertible, based on its derivative.

      Solution:

$$f'(x) = \frac{(5 + 2^x)(2^x \ln 2) - 2^x(2^x \ln 2)}{(5 + 2^x)^2}$$

$$= \frac{5 \cdot (2^x \ln 2)}{(5 + 2^x)^2} > 0 \text{ for all } x$$

      So, f is $\boxed{\text{monotone increasing}}$ on $(-\infty, \infty)$, which implies that f is invertible on $(-\infty, \infty)$.

ii. Find the inverse function of $f(x) = \dfrac{2^x}{5 + 2^x}$ and express it in the form $f^{-1}(x)$.

Solution:

$$y = \frac{2^x}{5 + 2^x}$$

$$x = \frac{2^y}{5 + 2^y}$$

$$x(5 + 2^y) = 2^y$$

$$5x + x \cdot 2^y = 2^y$$

$$2^y(x - 1) = -5x$$

$$2^y(1 - x) = 5x$$

$$2^y = \frac{5x}{1 - x}$$

$$y = f^{-1}(x) = \boxed{\log_2\left(\frac{5x}{1 - x}\right)}$$

(b) Consider the function $g(x) = 4e^{x-1} - 3e^{5-x^2}$, which is invertible on $[0, \infty)$. Find an equation of the line that is tangent to the curve $y = g^{-1}(x)$ at the point $(e, 2)$.

Hint: Do not attempt to explicitly identify the function $g^{-1}(x)$.

Solution:

The formula provided on the exam's cover page indicates that $(g^{-1})'(e) = \dfrac{1}{g'(2)}$.

$$\begin{aligned} g'(x) &= 4e^{x-1} - 3e^{5-x^2} \cdot (-2x) \\ &= 4e^{x-1} + 6xe^{5-x^2} \end{aligned}$$

$$\begin{aligned} g'(2) &= 4e^{2-1} + (6)(2)e^{5-2^2} \\ &= 4e + 12e = 16e \end{aligned}$$

So, $(g^{-1})'(e) = \dfrac{1}{g'(2)} = \dfrac{1}{16e}$.

Therefore, the line that is tangent to the curve $y = g^{-1}(x)$ at the point $(e, 2)$ can be expressed as

$$\boxed{y - 2 = \frac{1}{16e}(x - e)}$$

3. (25 pts) Parts (a) and (b) are not related.

(a) Use properties of logarithms to find $h'(x)$ for the function $h(x) = \ln\left(\frac{x^5\sqrt{\cos x}}{(2x^2+3x+9)^3}\right)$.

You do not need to fully simplify your answer.

Solution:

$$\begin{aligned}
h(x) &= \ln\left(\frac{x^5\sqrt{\cos x}}{(2x^2+3x+9)^3}\right) \\
&= \ln\left(x^5\right) + \ln\left((\cos x)^{1/2}\right) - \ln\left((2x^2+3x+9)^3\right) \\
&= 5\ln x + \frac{1}{2}\ln(\cos x) - 3\ln(2x^2+3x+9)
\end{aligned}$$

$$\begin{aligned}
h'(x) &= 5\cdot\frac{1}{x} + \frac{1}{2}\cdot\frac{-\sin x}{\cos x} - 3\cdot\frac{4x+3}{2x^2+3x+9} \\
&= \boxed{\frac{5}{x} - \frac{\sin x}{2\cos x} - \frac{3(4x+3)}{2x^2+3x+9}}
\end{aligned}$$

(b) Find the value of $y'(1)$ for the function $y = (1+2x)^{1/x}$.

Solution:

Logarithmic differentiation is applied. Note that $1 + 2x > 0$ on the interval $(-1/2, \infty)$ and y' is to be evaluated at $x = 1$, which lies on $(-1/2, \infty)$. Therefore, there is no need to take the absolute value of each side prior to taking the natural logarithm of each side.

$$\begin{aligned}
y &= (1+2x)^{1/x} \\
\ln y &= \ln\left((1+2x)^{1/x}\right) \\
\ln y &= \frac{1}{x}\ln(1+2x) \\
\frac{d}{dx}[\ln y] &= \frac{d}{dx}\left[\frac{\ln(1+2x)}{x}\right] \\
\frac{y'}{y} &= \frac{x \cdot \frac{2}{1+2x} - \ln(1+2x)}{x^2} \\
\frac{y'}{y} &= \frac{2}{x(1+2x)} - \frac{\ln(1+2x)}{x^2} \\
y' &= (1+2x)^{1/x}\left[\frac{2}{x(1+2x)} - \frac{\ln(1+2x)}{x^2}\right]
\end{aligned}$$

Therefore,

$$\begin{aligned}
y'(1) &= (1+2\cdot 1)^{1/1}\left[\frac{2}{1\cdot(1+2\cdot 1)} - \frac{\ln(1+2\cdot 1)}{1^2}\right] \\
&= 3\cdot\left(\frac{2}{3} - \ln 3\right) = \boxed{2 - 3\ln 3}
\end{aligned}$$

4. (36 pts) Parts (a), (b), and (c) are not related.

(a) Evaluate $\displaystyle\int \frac{6x+12}{x^2+4x+1}\,dx$.

Solution:

Apply u-substitution with $u = x^2+4x+1$.

$$u = x^2 + 4x + 1$$

$$\frac{du}{dx} = 2x + 4$$

$$du = 2(x+2)dx$$

Therefore,

$$\begin{aligned}
\int \frac{6x+12}{x^2+4x+1}\,dx &= 3\int \frac{2(x+2)dx}{x^2+4x+1} \\
&= 3\int \frac{du}{u} \\
&= 3\ln|u| + C \\
&= \boxed{3\ln|x^2+4x+1| + C}
\end{aligned}$$

(b) Evaluate $\int \frac{7^{\sqrt{x}}}{\sqrt{x}}\,dx$

Solution:

Apply u-substitution with $u = \sqrt{x}$.

$$u = x^{1/2}$$

$$\frac{du}{dx} = \frac{1}{2}x^{-1/2}$$

$$du = \frac{1}{2\sqrt{x}}\,dx$$

Therefore,

$$\begin{aligned}
\int \frac{7^{\sqrt{x}}}{\sqrt{x}}\,dx &= 2\int \frac{7^{\sqrt{x}}}{2\sqrt{x}}\,dx \\
&= 2\int 7^u\,du \\
&= 2\cdot\frac{7^u}{\ln 7} + C \\
&= \boxed{2\cdot\frac{7^{\sqrt{x}}}{\ln 7} + C}
\end{aligned}$$

(c) Evaluate $\displaystyle\int_{\pi/6}^{\pi/3} \tan\theta \, d\theta$. Simplify your answer to include only one logarithmic term.

Solution:

$$\int_{\pi/6}^{\pi/3} \tan\theta \, d\theta = \int_{\pi/6}^{\pi/3} \frac{\sin\theta}{\cos\theta} \, d\theta$$

Apply u-substitution with $u = \cos\theta$.

$$\begin{aligned} u &= \cos\theta \\ \frac{du}{d\theta} &= -\sin\theta \\ du &= -\sin\theta \, d\theta \end{aligned}$$

Also,

$$\begin{aligned} \theta = \pi/6 \quad &\Rightarrow \quad u = \cos(\pi/6) = \sqrt{3}/2 \\ \theta = \pi/3 \quad &\Rightarrow \quad u = \cos(\pi/3) = 1/2 \end{aligned}$$

Therefore,

$$\begin{aligned} \int_{\pi/6}^{\pi/3} \frac{\sin\theta}{\cos\theta} \, d\theta &= -\int_{\sqrt{3}/2}^{1/2} \frac{du}{u} \\ &= -\ln|u| \Big|_{\sqrt{3}/2}^{1/2} \\ &= \ln|u| \Big|_{1/2}^{\sqrt{3}/2} \\ &= \ln\left(\frac{\sqrt{3}}{2}\right) - \ln\left(\frac{1}{2}\right) \\ &= \ln\left(\frac{\sqrt{3}/2}{1/2}\right) \\ &= \boxed{\ln(\sqrt{3})} \\ &= \ln(3^{0.5}) = \boxed{\frac{\ln 3}{2}} \end{aligned}$$