Write your name and section number below. This exam has 5 problems and is worth 100 points. On each problem, you must show all your work to receive credit on that problem. You are allowed to use one handwritten 3 x 5 inch notecard (front and back) on this exam. You are NOT allowed to use any other notes, books, calculators, or electronic devices.

After you finish the exam, go to the designated area of the room to scan and upload your exam to Gradescope. **Please be sure to match your work with the corresponding problem.** Do not leave the room until you verify that your exam has been correctly uploaded.

Name:

Instructor/Section (Dougherty-001, Mitchell-002, Becker-003):

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1. For each of the following, **provide a short proof or justification**.

   (a) (8 points) Suppose A is an $n \times n$ matrix. Is $ker(A) \subset ker(A^2)$? Show this is true or provide a counterexample that shows it is false.

   (b) (8 points) Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ all be vectors in the same inner product space. If $\langle \mathbf{u}, \mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle$ is it necessarily true that $\mathbf{u} = \mathbf{v}$?

   (c) (8 points) If K is a positive definite matrix, then is K^2 also positive definite?

   (d) (8 points)

   i. Show that for all vectors $\mathbf{x}$ and $\mathbf{y}$ in an inner product space V,

   $$\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2 = 2\left(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2\right)$$

   using the norm induced by the inner product.

   ii. We mentioned in class that the p-norm, l^p in $\mathbb{R}^n$, only comes from an inner product if $p = 2$. Let's look at this in the case of $p = 1$. Use the identity from part i to show that the 1-norm, l^1, does not come from an inner product.

Solution:

(a) Yes. If $\mathbf{x} \in \ker(A)$ then $A\mathbf{x} = \mathbf{0}$, so we have

$$\begin{aligned} A^2\mathbf{x} &= A(A\mathbf{x}) \\ &= A\mathbf{0} \\ &= \mathbf{0} \end{aligned}$$

(b) No. Take for example

$$\mathbf{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \mathbf{w} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

with the usual dot product as the inner product. Then we have

$$\begin{aligned} &\langle \mathbf{u}, \mathbf{w} \rangle = 1 \\ &\langle \mathbf{v}, \mathbf{w} \rangle = 1 \\ &\mathbf{u} \neq \mathbf{v} \end{aligned}$$

(c) Yes, K^2 is also positive definite. First, we note that since K is symmetric, K^2 is also symmetric. Second, we look at:

$$\begin{aligned} \mathbf{x}^T K^2 \mathbf{x} &= \mathbf{x}^T K K \mathbf{x} \\ &= \mathbf{x}^T K^T K \mathbf{x} \\ &= (K\mathbf{x})^T (K\mathbf{x}) \\ &= \|K\mathbf{x}\|^2 \end{aligned}$$

This is greater than 0 for $K\mathbf{x} \neq \mathbf{0}$, and since K is positive definite, we know that $K\mathbf{x} \neq \mathbf{0}$. Therefore $K^2 > 0$.

(d) i. We calculate the norm squared of both vectors using its inner product:

$$\begin{aligned}\|\mathbf{x}+\mathbf{y}\|^2+\|\mathbf{x}-\mathbf{y}\|^2 &= \langle \mathbf{x}+\mathbf{y},\mathbf{x}+\mathbf{y}\rangle + \langle \mathbf{x}-\mathbf{y},\mathbf{x}-\mathbf{y}\rangle \\ &= \langle \mathbf{x},\mathbf{x}\rangle + 2\langle \mathbf{x},\mathbf{y}\rangle + \langle \mathbf{y},\mathbf{y}\rangle + \langle \mathbf{x},\mathbf{x}\rangle - 2\langle \mathbf{x},\mathbf{y}\rangle + \langle \mathbf{y},\mathbf{y}\rangle \\ &= 2\langle \mathbf{x},\mathbf{x}\rangle + 2\langle \mathbf{y},\mathbf{y}\rangle \\ &= 2\left(\|\mathbf{x}\|^2+\|\mathbf{y}\|^2\right)\end{aligned}$$

ii. Many pairs of vectors work as counterexamples. Take for example on $\mathbb{R}^2$,

$$\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\mathbf{y} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Then we have

$$\mathbf{x}+\mathbf{y} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\mathbf{x}-\mathbf{y} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Using the 1 norm, the left hand side equals

$$\begin{aligned}\|\mathbf{x}+\mathbf{y}\|^2+\|\mathbf{x}-\mathbf{y}\|^2 &= 2^2+2^2 \\ &= 8\end{aligned}$$

while the right hand side equals

$$\begin{aligned}2\left(\|\mathbf{x}\|^2+\|\mathbf{y}\|^2\right) &= 2(1^2+1^2) \\ &= 4\end{aligned}$$

Since these are not equal, the 1 norm does not have an associated inner poduct.

2. (16 points)

Consider $\mathbb{R}^{2\times 2}$, the vector space of 2×2 matrices with real entries, and the set of matrices:

$$\left\{ \begin{pmatrix} 2 & -3 \\ -1 & -3 \end{pmatrix}, \begin{pmatrix} 2 & -2 \\ 2 & -1 \end{pmatrix}, \begin{pmatrix} 4 & -5 \\ 1 & -4 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -3 & -2 \end{pmatrix} \right\}$$

(a) (10 points) Do these matrices span $\mathbb{R}^{2\times 2}$?

(b) (4 points) Find a basis for the span of these matrices.

(c) (2 points) What is the dimension of their span?

Solution:

(a) (10 points) Do these matrices span $\mathbb{R}^{2\times 2}$?

No, these matrices do not span $\mathbb{R}^{2\times 2}$. We can either see if we can solve for *every* right-hand-side $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, or equivalently see if we have a non-trivial null-space. That is, since we know $\mathbb{R}^{2\times 2}$ has dimension 4, and since we have 4 vectors in our set, whether they span the space is linked to whether they are linearly independent.
So, we could look for non-zero solutions to

$$c_1 \begin{pmatrix} 2 & -3 \\ -1 & -3 \end{pmatrix} + c_2 \begin{pmatrix} 2 & -2 \\ 2 & -1 \end{pmatrix} + c_3 \begin{pmatrix} 4 & -5 \\ 1 & -4 \end{pmatrix} + c_4 \begin{pmatrix} 0 & -1 \\ -3 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

This yields the homogeneous system:

$$\begin{aligned} 2c_1 + 2c_2 + 4c_3 + 0c_4 &= 0 \\ -3c_1 - 2c_2 - 5c_3 - c_4 &= 0 \\ -c_1 + 2c_2 + c_3 - 3c_4 &= 0 \\ -3c_1 - c_2 - 4c_3 - 2c_4 &= 0 \end{aligned}$$

We find the solutions by calculating the REF of the coefficient matrix:

$$\left(\begin{array}{cccc|c} 2 & 2 & 4 & 0 & 0 \\ -3 & -2 & -5 & -1 & 0 \\ -1 & 2 & 1 & -3 & 0 \\ -3 & -1 & -4 & -2 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 2 & 2 & 4 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 3 & 3 & -3 & 0 \\ 0 & 2 & 2 & -2 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 2 & 2 & 4 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

The homogeneous system has two free variables, so there are infinitely many solutions. We conclude that these four matrices do not form a basis of $\mathbb{R}^{2\times 2}$. *Note: your coefficient matrix might look different, e.g., permuting rows or columns. Those are valid too; no matter how you do it, you should find that there are 2 free variables.*

(b) (4 points) Find a basis for the span of these matrices.

The first and second columns of our coefficient matrix are pivot columns, so the first and second matrix of our set form the basis for the span:

$$\left\{ \begin{pmatrix} 2 & -3 \\ -1 & -3 \end{pmatrix}, \begin{pmatrix} 2 & -2 \\ 2 & -1 \end{pmatrix} \right\}.$$

Note: The above basis is the one you'd find following the systematic techniques we taught and if you setup the coefficient matrix the way we did, but in fact any two of the matrices from that set form a basis for this problem.

(c) (2 points) What is the dimension of their span?

As there are two basis vectors, the dimension of the span is 2.

3. (20 points) Suppose matrix $A = \begin{pmatrix} 0 & 1 & -2 \\ 4 & -6 & 0 \end{pmatrix}$

   (a) (15 points) Find a basis (and the dimension) for each of the four fundamental subspaces of A.

   (b) (5 points) Are $\ker(A)$ and $\operatorname{coimg}(A)$ complementary subspaces? Explain. Recall that two subspaces, W and Z, of a vector space V are complementary if (a) $W \cap Z = \{0\}$ and (b) $W + Z = V$. (Hint: You don't have to use the definition directly. Can you use the basis for $\ker(A)$ and $\operatorname{coimg}(A)$?)

Solution:

(a) We first find the REF of the matrix:

$$\begin{pmatrix} 0 & 1 & -2 \\ 4 & -6 & 0 \end{pmatrix} \to \begin{pmatrix} 4 & -6 & 0 \\ 0 & 1 & -2 \end{pmatrix}$$

Columns one and two are pivot columns, so the **image** has dimension 2 and we select the first and second columns of the original matrix as our basis for the image:

$$\left\{ \begin{pmatrix} 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ -6 \end{pmatrix} \right\} \text{ is a basis for } \operatorname{Im}(A).$$

Note that $\operatorname{img}(A) = \mathbb{R}^2$. The techniques we taught in class lead to the basis above, but any two linearly independent vectors in $\mathbb{R}^2$ would form a basis. In particular, any two columns from A actually form a basis.

The **coimage** also has dimension 2 and we select the first and second column of the transpose of the REF for the basis vectors:

$$\left\{ \begin{pmatrix} 4 \\ -6 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \right\} \text{ is a basis for } \operatorname{CoImg}(A).$$

There is one free variable (evident from the REF of the original matrix), so the **kernel** has dimension 1. We find our basis vector by solving the homogeneous equation:

$$\begin{pmatrix} 4 & -6 & 0 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \text{ i.e., } \left(\begin{array}{ccc|c} 4 & -6 & 0 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right)$$

so $x_3 = c$ for any $c \in \mathbb{R}$, $x_2 = 2x_3 = 2c$ and $x_1 = \frac{6}{4}x_2 = 3c$, i.e,

$$\left\{ \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \right\} \text{ is a basis for } \operatorname{Ker}(A).$$

Finally, to calculate the **cokernel**, we must find the REF of A^T:

$$A^T = \begin{pmatrix} 0 & 4 \\ 1 & -6 \\ -2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -6 \\ 0 & 4 \\ -2 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -6 \\ 0 & 4 \\ 0 & -12 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -6 \\ 0 & 4 \\ 0 & 0 \end{pmatrix}$$

A^T is full column rank, so its kernel is just the zero vector:

$$\operatorname{coker}(A) = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

and the only basis for this trivial vector space is the empty set:

$\emptyset = \{\}$ is the only basis of $\operatorname{CoKer}(A)$.

Note that the set containing zero $\{\mathbf{0}\}$ is not the empty set, and is acutally a linearly dependent set and cannot be a basis of any vector space.

In this case, we say the dimension is 0. A shortcut: recalling that the dimension of the cokernel plus the dimension of the image must equal the number of rows, then since we have 2 rows and know the dimension of the image is 2, we can infer without calculation that the cokernel has dimension zero, hence must be the trivial vector space of just $\mathbf{0}$.

(b) Yes, they are. We confirm this by showing the matrix with all three basis vectors for columns is non-singular (that is, the 2 basis vectors for the coimg, and the 1 basis vector for the kernel). This proves that the vectors are all linearly independent and span $\mathbb{R}^3$. There are several ways to show it is non-singular; one way is to check that it has nonzero determinant:

$$\det \begin{pmatrix} 4 & 0 & 3 \\ -6 & 1 & 2 \\ 0 & -2 & 1 \end{pmatrix} = 4 \det \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} + 3 \det \begin{pmatrix} -6 & 1 \\ 0 & -2 \end{pmatrix}$$

$$= 4(5) + 3(12)$$

$$= 56$$

$$\neq 0$$

Recall we discussed complementary subspaces in problem 2.2.24 on homework 4

4. For each of following operators, if they are linear find their matrix representation (i.e., find a matrix A such that $L(\mathbf{x}) = A\mathbf{x}$), otherwise find a specific counter example (using numbers/vectors, not variables) that proves they are non-linear.

   (a) (4 points) $L : \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x - y \\ z + 2x + 3y \end{bmatrix}$$

   (b) (4 points) $L : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} |x + y| \\ 0 \end{bmatrix}$$

   (c) (4 points) $L : \mathbb{R}^4 \to \mathbb{R}^3$ given by

$$L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_1 x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

   (d) (4 points) $L : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + y + 1 \\ x - y - 1 \end{bmatrix}$$

Solution:

(a) $L : \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x - y \\ z + 2x + 3y \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & -1 & 0 \\ 2 & 3 & 1 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

(b) $L : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} |x + y| \\ 0 \end{bmatrix}$$

is not linear. One example to show this (there are many) is

$$L\left(-1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = 1 \neq -1 = -1L\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right).$$

(c) $L : \mathbb{R}^4 \to \mathbb{R}^3$ given by

$$L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_1 x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

is not linear. One example to show this (there are many) is

$$L\left(2\begin{bmatrix}1\\1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}4\\0\\0\end{bmatrix}\neq\begin{bmatrix}2\\0\\0\end{bmatrix}=2L\left(\begin{bmatrix}1\\1\\0\\0\end{bmatrix}\right).$$

(d) $L:\mathbb{R}^2\to\mathbb{R}^2$ given by

$$L\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}x+y+1\\x-y-1\end{bmatrix}$$

is not linear. One example to show this (there are many) is

$$L(\mathbf{0})=\begin{bmatrix}1\\-1\end{bmatrix}\neq\mathbf{0}.$$

5. The following two problems are unrelated

   (a) (6 points) Let $f, g \in C^0[a,b]$ for some real numbers $a < b$, and define $\langle f, g\rangle = \int_a^b f(x)g(x)\,dx$. If $f(x) \equiv 1$ and $g(x) = x$ and $a = -1$, is there a value for b for which f and g are orthogonal? If so, which value? If not, explain why not.

   (b) (10 points) Let $\|\mathbf{v}\|_a$ and $\|\mathbf{v}\|_b$ both be norms on a vector space V. Prove that $\|\mathbf{v}\| = \max\{\|\mathbf{v}\|_a, \|\mathbf{v}\|_b\}$ is a valid norm on V.

Solution:

(a) Yes, choosing the value of $b = 1$ will make f and g orthogonal, since $\langle f, g\rangle = \int_{-1}^1 x\,dx = 0$ (you can either do the integral, or observe that it's an odd function). *Note: about a third of the class concluded there was no value of $b > -1$ that worked because they did the integral wrong, getting a value of $\frac{1}{2}b^2 + \frac{1}{2}$ rather than a value of $\frac{1}{2}b^2 - \frac{1}{2}$. Please brush up on your calculus!*

(b) We need to check the 3 properties required of the norms: (1) positivity, (2) homogeneity, and (3) triangle inequality.

To check (1), let's split it up as (1a): show $\|\mathbf{v}\| \geq 0$ for all $\mathbf{v}$, and (1b): show $\|\mathbf{v}\| = 0 \implies \mathbf{v} = \mathbf{0}$. For (1a), this follows since both $\|\mathbf{v}\|_a \geq 0$ and $\|\mathbf{v}\|_b \geq 0$ since those are norms, hence their max is also non-negative.

For (1b), if $\|\mathbf{v}\| = 0$ then we know in particular $\|\mathbf{v}\|_a = 0$, so since $\|\mathbf{v}\|_a$ is a norm, we concluded $\mathbf{v} = \mathbf{0}$.

To check (2), for any $c \in \mathbb{R}$, compute $\|c\mathbf{v}\| = \max\{\|c\mathbf{v}\|_a, \|c\mathbf{v}\|_b\} = \max\{|c| \cdot \|\mathbf{v}\|_a, |c| \cdot \|\mathbf{v}\|_b\} = |c| \max\{\|\mathbf{v}\|_a, \|\mathbf{v}\|_b\} = \|c\| \cdot \|\mathbf{v}\|$.

To check (3), take any $\mathbf{v}, \mathbf{w} \in V$, and compute $\|\mathbf{v}+\mathbf{w}\| = \max\{\|\mathbf{v}+\mathbf{w}\|_a, \|\mathbf{v}+\mathbf{w}\|_b\} \leq \max\{\|\mathbf{v}\|_a + \|\mathbf{w}\|_a, \|\mathbf{v}\|_b + \|\mathbf{w}\|_b\} \leq \max\{\|\mathbf{v}\|_a + \|\mathbf{v}\|_b\} + \max\{\|\mathbf{w}\|_a + \|\mathbf{w}\|_b\} = \|\mathbf{v}\| + \|\mathbf{w}\|$.

Note: some students said that since $\|\mathbf{v}\|$ will always equal either $\|\mathbf{v}\|_a$ or $\|\mathbf{v}\|_b$, and since those are valid norms, then the result follows. The issue is that which one it equals depends on $\mathbf{v}$ and that can mess things up. In particular, if we took $\|\mathbf{v}\| = \min\{\|\mathbf{v}\|_a, \|\mathbf{v}\|_b\}$, it's still true that $\|\mathbf{v}\|$ will always equal either $\|\mathbf{v}\|_a$ or $\|\mathbf{v}\|_b$, but if we used the min *then it's not a valid norm!*

Note: some students confused these with the p-norms. They are not a p-norm.