Write your name and section number below. This exam has 5 problems and is worth 100 points. On each problem, you must show all your work to receive credit on that problem. You are allowed to use one handwritten 3 x 5 inch notecard (front and back) on this exam. You are NOT allowed to use any other notes, books, calculators, or electronic devices.

After you finish the exam, go to the designated area of the room to scan and upload your exam to Gradescope. **Please be sure to match your work with the corresponding problem.** Do not leave the room until you verify that your exam has been correctly uploaded.

Name:

Instructor/Section (Dougherty-001, Mitchell-002, Becker-003):

---

1. For each of the following, **provide a short proof or justification**.

   (a) (8 points) Suppose A is an $n \times n$ matrix. Is $ker(A) \subset ker(A^2)$? Show this is true or provide a counterexample that shows it is false.

   (b) (8 points) Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ all be vectors in the same inner product space. If $\langle \mathbf{u}, \mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle$ is it necessarily true that $\mathbf{u} = \mathbf{v}$?

   (c) (8 points) If K is a positive definite matrix, then is K^2 also positive definite?

   (d) (8 points)

      i. Show that for all vectors $\mathbf{x}$ and $\mathbf{y}$ in an inner product space V,

      $$\|\mathbf{x} + \mathbf{y}\|^2 + \|\mathbf{x} - \mathbf{y}\|^2 = 2\left(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2\right)$$

      using the norm induced by the inner product.

      ii. We mentioned in class that the p-norm, l^p in $\mathbb{R}^n$, only comes from an inner product if $p = 2$. Let's look at this in the case of $p = 1$. Use the identity from part i to show that the 1-norm, l^1, does not come from an inner product.

2. (16 points)

   Consider $\mathbb{R}^{2\times 2}$, the vector space of 2×2 matrices with real entries, and the set of matrices:

$$\left\{ \begin{pmatrix} 2 & -3 \\ -1 & -3 \end{pmatrix}, \begin{pmatrix} 2 & -2 \\ 2 & -1 \end{pmatrix}, \begin{pmatrix} 4 & -5 \\ 1 & -4 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -3 & -2 \end{pmatrix} \right\}$$

   (a) (10 points) Do these matrices span $\mathbb{R}^{2\times 2}$?

   (b) (4 points) Find a basis for the span of these matrices.

   (c) (2 points) What is the dimension of their span?

3. (20 points) Suppose matrix $A = \begin{pmatrix} 0 & 1 & -2 \\ 4 & -6 & 0 \end{pmatrix}$

   (a) (15 points) Find a basis (and the dimension) for each of the four fundamental subspaces of A.

   (b) (5 points) Are $\ker(A)$ and $\text{coimg}(A)$ complementary subspaces? Explain. Recall that two subspaces, W and Z, of a vector space V are complementary if (a) $W \cap Z = \{0\}$ and (b) $W + Z = V$. (Hint: You don't have to use the definition directly. Can you use the basis for $\ker(A)$ and $\text{coimg}(A)$?)

4. For each of following operators, if they are linear find their matrix representation (i.e., find a matrix A such that $L(\mathbf{x}) = A\mathbf{x}$), otherwise find a specific counter example (using numbers/vectors, not variables) that proves they are non-linear.

   (a) (4 points) $L : \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x - y \\ z + 2x + 3y \end{bmatrix}$$

   (b) (4 points) $L : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} |x + y| \\ 0 \end{bmatrix}$$

   (c) (4 points) $L : \mathbb{R}^4 \to \mathbb{R}^3$ given by

$$L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_1 x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

   (d) (4 points) $L : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + y + 1 \\ x - y - 1 \end{bmatrix}$$

5. The following two problems are unrelated

   (a) (6 points) Let $f, g \in C^0[a, b]$ for some real numbers $a < b$, and define $\langle f, g \rangle = \int_a^b f(x)g(x)\,dx$. If $f(x) \equiv 1$ and $g(x) = x$ and $a = -1$, is there a value for b for which f and g are orthogonal? If so, which value? If not, explain why not.

   (b) (10 points) Let $\|\mathbf{v}\|_a$ and $\|\mathbf{v}\|_b$ both be norms on a vector space V. Prove that $\|\mathbf{v}\| = \max\{\|\mathbf{v}\|_a, \|\mathbf{v}\|_b\}$ is a valid norm on V.