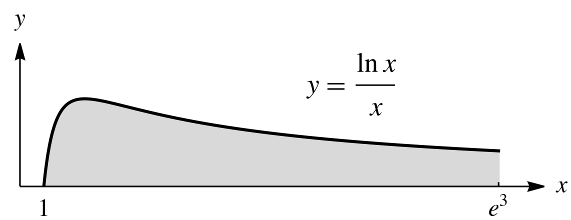
1. Region $\mathcal{R}$ is bounded by the curve $y = (\ln x)/x$ and the x-axis on the interval $\left[1, e^3\right]$.



(a) (24 pts) Set up (but <u>do not evaluate</u>) integrals to find the following quantities.

i. The volume of the solid generated by rotating region $\mathcal{R}$ about the y-axis.

ii. The volume of the solid with $\mathcal{R}$ as the base and cross-sections perpendicular to the x-axis in the shape of squares.

iii. The area of the surface generated by rotating the given curve $y = (\ln x)/x$ on $\left[1, e^3\right]$ about the line $y = -2$.

Solution:

i. By the Shell Method,

$$V = \int_a^b 2\pi r h\, dx = \boxed{\int_1^{e^3} 2\pi x \cdot \frac{\ln x}{x}\, dx} = \boxed{\int_1^{e^3} 2\pi \ln x\, dx}.$$

ii. Each square has a side length of $(\ln x)/x$ and area of $A(x) = ((\ln x)/x)^2$. Therefore the volume of the solid is

$$V = \int_a^b A(x)\, dx = \boxed{\int_1^{e^3} \left(\frac{\ln x}{x}\right)^2 dx}.$$

iii. The derivative of $y = \dfrac{\ln x}{x}$ is $y' = \dfrac{1 - \ln x}{x^2}$ and the radius r is $\dfrac{\ln x}{x} + 2$, so the surface area is

$$S = \int_a^b 2\pi r\, ds = \int_a^b 2\pi r \sqrt{1 + (y')^2}\, dx = \boxed{\int_1^{e^3} 2\pi \left(\frac{\ln x}{x} + 2\right) \sqrt{1 + \left(\frac{1 - \ln x}{x^2}\right)^2}\, dx}.$$

(b) (18 pts) A thin metal plate with constant density ρ covers the same region $\mathcal{R}$ shown. The moment about the y-axis of the plate is $M_y = 3 + 6e^3$. Evaluate integrals to solve the following problems.

i. Find the value of ρ.

ii. Find the mass of the plate.

Solution:

i.

$$\begin{aligned} M_y &= \int_a^b \rho\, x f(x)\, dx \\ &= \int_1^{e^3} \rho\, x \cdot \frac{\ln x}{x}\, dx \\ &= \rho \int_1^{e^3} \ln x\, dx \end{aligned}$$

Apply Integration by Parts with $u = \ln x,\ \ du = dx/x$ and $dv = dx,\ \ v = x$.

$$\begin{aligned} &= \rho\left([x \ln x]_1^{e^3} - \int_1^{e^3} dx \right) \\ &= \rho\left(\left(e^3 \ln e^3 - \ln 1\right) - [x]_1^{e^3} \right) \\ &= \rho\left(\left(3e^3 - 0\right) - \left(e^3 - 1\right) \right) \\ &= \rho\left(2e^3 + 1\right) \end{aligned}$$

It is given that $M_y = 3 + 6e^3$, so $\rho = \boxed{3}$.

ii.

$$\begin{aligned} M &= \rho \int_a^b f(x)\, dx \\ &= 3 \int_1^{e^3} \frac{\ln x}{x}\, dx \end{aligned}$$

Let $u = \ln x,\ \ du = dx/x$. Then the bounds $x = 1$ to e^3 convert to $u = 0$ to 3.

$$\begin{aligned} &= 3 \int_0^3 u\, du \\ &= 3 \left[\frac{u^2}{2} \right]_0^3 \\ &= \frac{3}{2}(9 - 0) = \boxed{\frac{27}{2}} \end{aligned}$$

2. (12 pts) Solve the differential equation for y.

$$\sec^4 x\,\frac{dy}{dx} = \frac{\sin^3 x}{y}$$

Solution: This is a separable equation.

$$\begin{aligned}
\sec^4 x\,\frac{dy}{dx} &= \frac{\sin^3 x}{y}\\
\int y\,dy &= \int \sin^3 x \cos^4 x\,dx\\
\frac{y^2}{2} &= \int \sin^2 x \cos^4 x \sin x\,dx
\end{aligned}$$

Let $u = \cos x,\ \ du = -\sin x\,dx$. Substitute $\sin^2 x = 1 - \cos^2 x$.

$$\begin{aligned}
\frac{y^2}{2} &= \int \underbrace{\left(1-\cos^2 x\right)}_{1-u^2}\underbrace{\cos^4 x}_{u^4}\underbrace{\sin x\,dx}_{-du}\\
&= \int -\left(1-u^2\right)u^4\,du\\
&= \int \left(u^6 - u^4\right)du\\
&= \frac{u^7}{7} - \frac{u^5}{5} + C\\
&= \frac{1}{7}\cos^7 x - \frac{1}{5}\cos^5 x + C
\end{aligned}$$

$$\boxed{y = \pm\sqrt{2\left(\frac{1}{7}\cos^7 x - \frac{1}{5}\cos^5 x\right) + C}}$$

3. The following parts are not related. <u>Justify</u> all answers.

   (a) (8 pts) Does the sequence $\{m\arctan(5/m)\}$ converge? If so, what does it converge to?

   (b) (14 pts) Find the sum of the series $\displaystyle\sum_{n=1}^{\infty}\frac{6}{n^2+3n+2}$ or explain why the sum does not exist.

   (*Hint:* Begin with a partial fraction decomposition.)

Solution:

(a) The limit has the form of an indeterminate product $\infty \cdot 0$. Rewrite the product as a quotient, then apply L'Hopital's Rule.

$$\lim_{m\to\infty} m\arctan(5/m) = \lim_{m\to\infty}\frac{\arctan(5/m)}{1/m} \overset{LH}{=} \lim_{m\to\infty}\frac{\frac{1}{1+25/m^2}\left(-\frac{5}{m^2}\right)}{-\frac{1}{m^2}} = \lim_{m\to\infty}\frac{5}{1+\frac{25}{m^2}} = 5,$$

so the sequence converges to $\boxed{5}$.

(b) The partial fraction decomposition of $\dfrac{6}{n^2+3n+2}$ has the form

$$\frac{6}{(n+1)(n+2)} = \frac{A}{n+1} + \frac{B}{n+2}.$$

Solving $A(n+2)+B(n+1)=6$ gives $A=6$ and $B=-6$, so the series can be written as

$$\sum_{n=1}^{\infty} \frac{6}{(n+1)(n+2)} = \sum_{n=1}^{\infty} \left(\frac{6}{n+1} - \frac{6}{n+2}\right).$$

The series is telescoping with partial sum

$$s_n = \left(\frac{6}{2} - \cancel{\frac{6}{3}}\right) + \left(\cancel{\frac{6}{3}} - \cancel{\frac{6}{4}}\right) + \left(\cancel{\frac{6}{4}} - \cancel{\frac{6}{5}}\right) + \cdots + \left(\cancel{\frac{6}{n+1}} - \frac{6}{n+2}\right) = 3 - \frac{6}{n+2}.$$

The sum of the series equals $\lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty}\left(3 - \dfrac{6}{n+2}\right) = 3 - 0 = \boxed{3}$.

4. The following parts are not related. Justify all answers.

(a) (8 pts) Let s_n be the nth partial sum of the series $\sum\limits_{n=1}^{\infty} a_n$. Suppose $\lim\limits_{n\to\infty} s_n = 8$.

i. Find $\lim\limits_{n\to\infty} a_n$.

ii. Find the sum of the series.

Solution:

i. Because the limit of the partial sums exists, the series converges and the sequence a_n converges to 0. Therefore $\lim\limits_{n\to\infty} a_n = \boxed{0}$.

ii. The sum of the series equals the limit of the partial sums which is $\boxed{8}$.

(b) (8 pts) The nth partial sum of the series $\sum\limits_{n=1}^{\infty} b_n$ is $s_n = 5\left(\dfrac{2}{5}\right)^n$.

i. Find the third term of the series.

ii. Find the sum of the series.

Solution:

i. Because the partial sum $s_2 = b_1 + b_2$ and $s_3 = b_1 + b_2 + b_3$, the third term is

$$b_3 = s_3 - s_2 = 5\left(\frac{2}{5}\right)^3 - 5\left(\frac{2}{5}\right)^2 = \frac{8}{25} - \frac{4}{5} = \boxed{-\frac{12}{25}}.$$

ii. The sum of the series equals $\lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} 5\left(\dfrac{2}{5}\right)^n = \boxed{0}$ because s_n corresponds to an r^n sequence with $r = 2/5 < 1$.

(c) (8 pts) Find the value of k that satisfies

$$e^{2k} + e^{4k} + e^{6k} + e^{8k} + \cdots = \frac{1}{2}.$$

Solution: The series on the left side of the equation is geometric with first term $a = e^{2k}$ and ratio $r = e^{2k}$. The sum of the series is

$$S = \frac{a}{1-r} = \frac{e^{2k}}{1-e^{2k}}.$$

Set $S = 1/2$ and solve for k.

$$\begin{aligned} S = \frac{e^{2k}}{1-e^{2k}} &= \frac{1}{2}. \\ 2e^{2k} &= 1 - e^{2k} \\ e^{2k} &= \frac{1}{3} \\ 2k &= \ln\frac{1}{3} \\ k &= \boxed{\frac{1}{2}\ln\frac{1}{3}} = \boxed{-\frac{1}{2}\ln 3} \end{aligned}$$

Note that $r = e^{2k} = \frac{1}{3}$, so $|r| < 1$ and the series converges.