1. For each of the following limits, evaluate it or show that it does not exist.

   (a) (8 points) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3 - 4y^2}{x^2 + 2y^2}$

   (b) (8 points) $\displaystyle\lim_{(x,y)\to(5,3)} \frac{(2x - y - 2)^2 - (2x + y - 8)^2}{6 - 2y}$

   Solution:

   (a) This limit is a $\frac{0}{0}$-indeterminate form. We investigate distinct paths.
   Along the path $x = 0$, as $(x, y) = (0, y) \to (0, 0)$, we have

   $$\frac{x^3 - 4y^2}{x^2 + 2y^2} = \frac{-4y^2}{2y^2} = -2 \to -2.$$

   But, along the path $y = 0$, as $(x, y) = (x, 0) \to (0, 0)$, we have

   $$\frac{x^3 - 4y^2}{x^2 + 2y^2} = \frac{x^3}{x^2} = x \to 0.$$

   Since the function approaches different values as (x, y) approaches $(0, 0)$ along different paths, then we know the limit does not exist.

   (b) We note that this limit is a $\frac{0}{0}$-indeterminate form. We also note that the numerator is the difference of two squares and that

   $$(2x - y - 2) - (2x + y - 8) = 6 - 2y.$$

   So, we have

   $$\lim_{(x,y)\to(5,3)} \frac{(2x - y - 2)^2 - (2x + y - 8)^2}{6 - 2y}$$

   $$= \lim_{(x,y)\to(5,3)} \frac{[(2x - y - 2) - (2x + y - 8)][(2x - y - 2) + (2x + y - 8)]}{(2x - y - 2) - (2x + y - 8)}$$

   $$= \lim_{(x,y)\to(5,3)} (2x - y - 2) + (2x + y - 8)$$

   $$= \lim_{(x,y)\to(5,3)} 4x - 10 = 10.$$

2. Consider $f(x, y) = \ln(y(x - 1)^2)$.

   (a) (5 points) For which points (x, y) is $f(x, y)$ continuous?

   (b) (9 points) Determine the linear approximation (linearization) of $f(x, y)$ centered at $(2, e)$.

   (c) (5 points) Use your linear approximation from (b) to approximate $f\left(\frac{39}{20}, \frac{21e}{20}\right)$.

   (d) (9 points) Using Taylor's Theorem, what is the maximum possible error when using the linear approximation from (b) to approximate $f(x, y)$ when $|x - 2| \le 0.2$ and $|y - e| \le 0.2$? (You may find it useful in to recall that $e \approx 2.71$. You should **not** use this for earlier parts of the problem.)

Solution:

(a) We need the argument of the logarithm to be strictly greater than 0. Note that $(x-1)^2 > 0$ when $x \neq 1$, so this means we need $y > 0$, and the set on which this function is continuous is

$$\{(x, y) | x \neq 1, y > 0\}.$$

(b) We have

$$f_x = \frac{2y(x-1)}{y(x-1)^2} = \frac{2}{x-1}$$

and

$$f_y = \frac{(x-1)^2}{y(x-1)^2} = \frac{1}{y}.$$

So,

$$\begin{aligned} L(x, y) &= f(2, e) + f_x(2, e)(x-2) + f_y(2, e)(y-e) \\ &= 1 + 2(x-2) + \frac{1}{e}(y-e). \end{aligned}$$

(c)

$$\begin{aligned} f\left(\frac{39}{20}, \frac{21e}{20}\right) &\approx L\left(\frac{39}{20}, \frac{21e}{20}\right) \\ &= 1 + 2\left(\frac{39}{20} - 2\right) + \frac{1}{e}\left(\frac{21e}{20} - e\right) \\ &= 1 - \frac{1}{20} \\ &= \frac{19}{20} \end{aligned}$$

(d) We will use the formula

$$|E(x, y)| \leq \frac{M}{(n+1)!}(\triangle x + \triangle y)^{n+1}$$

where $n = 1$ (because we're using the order 1 Taylor polynomial) and M is an upper bound on the absolute value of the second partial derivatives.

The second partial derivatives are

$$f_{xx} = -\frac{2}{(x-1)^2} \qquad f_{xy} = 0 \qquad f_{yy} = -\frac{1}{y^2}.$$

For the region under consideration, we see

$$|f_{xx}| \leq \frac{2}{(1.8-1)^2} = \frac{2}{0.64} = \frac{1}{0.32},$$

$$|f_{xy}| = 0,$$

and

$$|f_{yy}| \leq \frac{1}{(e-0.2)^2} < \frac{1}{2.51^2} < \frac{1}{0.32}.$$

So, we will use $M = \frac{1}{0.32}$. Then, the error is at most

$$|E(x, y)| \leq \frac{1}{(0.32)(2)}(0.2 + 0.2)^2 = \frac{0.16}{0.64} = \frac{1}{4}.$$

3. The elevation of the ground in a park is given by $g(x,y) = 8xy - \frac{1}{4}(x+y)^4$.

   (a) (13 points) Determine the location of all local maximums, local minimums, and saddle points. (Assume the park has no boundary. And, be sure to fully classify each as a local maximum, local minimum, or a saddle point.)

   (b) (15 points) Sam the Squirrel is running through the park with a big acorn in his mouth and has location $\mathbf{r}(t) = \langle t^3,\ 3t - t^2\rangle$ after t minutes.

      i. Find the rate of change of Sam's elevation with respect to *time* when he is at the point $(1,2)$.

      ii. Find the rate of change of Sam's elevation with respect to *distance* when he is at the point $(1,2)$.

      iii. Sam accidentally drops his acorn at the point $(1,2)$ and it rolls in the direction of steepest descent. Find the unit vector in this direction.

Solution:

(a) We first need to locate any critical numbers of $g(x,y)$. So, we have

$$\begin{aligned} g_x &= 8y - (x+y)^3 = 0 \\ g_y &= 8x - (x+y)^3 = 0. \end{aligned}$$

It follows from this that $8x = (x+y)^3 = 8y$, which means $x = y$. If we look at the second equation, we now have

$$\begin{aligned} 8x - (2x)^3 &= 0 \\ 8x(1-x^2) &= 0 \\ x &= 0, \pm 1. \end{aligned}$$

So, the critical points are $(-1,-1)$, $(0,0)$, and $(1,1)$.
We will now apply the second derivative test. Note that

$$g_{xx} = -3(x+y)^2 = g_{yy} \qquad g_{xy} = 8 - 3(x+y)^2.$$

So,

$$D = g_{xx}g_{yy} - g_{xy}^2 = 9(x+y)^4 - (8 - 3(x+y)^2)^2.$$

We see that $D(0,0) < 0$, so $(0,0)$ is the location of a saddle point.
We have $D(1,1) = D(-1,-1) = 9(16) - 16 > 0$ and $g_{xx}(1,1) = g_{xx}(-1,-1) < 0$, so $(1,1)$ and $(-1,-1)$ are both the locations of local maximum values.

(b) The point $(1,2)$ corresponds to $\mathbf{r}(1)$. We note the following:

$$\begin{aligned} \mathbf{r}'(t) &= \langle 3t^2, 3-2t\rangle \\ \mathbf{r}'(1) &= \langle 3, 1\rangle \\ ||\mathbf{r}'(1)|| &= \sqrt{10} \\ \nabla g(x,y) &= \langle 8y - (x+y)^3, 8x - (x+y)^3\rangle \\ \nabla g(1,2) &= \langle -11, -19\rangle \\ ||\nabla g(1,2)|| &= \sqrt{(-11)^2 + (-19)^2} = \sqrt{121+361} = \sqrt{482}. \end{aligned}$$

i.

$$\frac{dg}{dt} = \nabla g(1,2) \cdot \mathbf{r}'(1) = -52.$$

ii.

$$\frac{dg}{ds} = \frac{\nabla g(1,2) \cdot \mathbf{r}'(1)}{||\mathbf{r}'(1)||} = -\frac{52}{\sqrt{10}}.$$

iii.

$$-\frac{\nabla g(1,2)}{||\nabla g(1,2)||} = \left\langle \frac{11}{\sqrt{482}}, \frac{19}{\sqrt{482}} \right\rangle.$$

4. (28 points) Pam the Penguin has bought a parcel of land that is the shape of a disk with radius 3 kilometers that can be described by the inequality $x^2 + y^2 \leq 9$. The average temperature on this disk, in degrees Celsius, is given by $T(x, y) = 10 + 2x^2 + (y - 1)^2$.

   (a) Pam wants to place a mailbox on the point on the boundary of the disk with the coldest average temperature.

      i. (11 points) Use Lagrange multipliers to determine which point on the boundary she should place her mailbox.

      ii. (2 points) What will the average temperature be at the point you found in i?

   (b) Pam wants to build her home at the point on the disk with the coldest average temperature.

      i. (6 points) Explain why the Extreme Value Theorem guarantees there is a point on this disk with the coldest average temperature.

      ii. (9 points) At what point on the disk should Pam build her home to ensure it has the coldest average temperature possible?

Solution:

(a) We note that $\nabla T = \langle 4x, 2(y-1)\rangle$. If we let $g(x, y) = x^2 + y^2$, then our constraint is given by $g(x, y) = 9$. We have $\nabla g = \langle 2x, 2y\rangle$. Using Lagrange multipliers, we need to solve the following system:

$$\begin{aligned} 4x &= 2\lambda x \\ 2(y-1) &= 2\lambda y \\ x^2 + y^2 &= 9 \end{aligned}$$

From the first equation, we have $(2 - \lambda)x = 0$, so $\lambda = 2$ or $x = 0$.
Case: $\lambda = 2$: The second equation yields $y = -1$. From this, the third equation yields $x = \pm 2\sqrt{2}$. So, we have two points from this case to consider, $(\pm 2\sqrt{2}, -1)$.

$$T(\pm 2\sqrt{2}, -1) = 30.$$

Case: $x = 0$: The third equation yields $y = \pm 3$. So, we have two points from this case to consider, $(0, \pm 3)$.

$$\begin{aligned} T(0,3) &= 14 \\ T(0,-3) &= 26. \end{aligned}$$

i. Pam should place her mailbox at $(0, 3)$.

ii. The temperature at Pam's mailbox will be 14 degrees Celsius.

(b) i. Since the disk is a closed and bounded set and $T(x,y)$ is continuous on the disk (because it is a polynomial), then the Extreme Value Theorem guarantees the existence of an absolute minimum value of $T(x,y)$ over the disk. That is, there is a coldest point on the disk.

ii. We first find any critical points by consider the equations

$$T_x = 4x = 0$$
$$T_y = 2(y-1) = 0.$$

The only solution, and therefore only critical point, is $(0,1)$, which does lie in the disk. We see that $T(0,1) = 10$, and this is less than any of the previously considered values on the boundary of the disk. So, Pam should build her house at $(0,1)$.