1. [2360/031225 (10 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given. Please write your answers in a single column separate from any work you do to arrive at the answer.

   (a) If $\mathbf{A}$ and $\mathbf{B}$ are both $n \times n$ diagonal matrices, then $\mathbf{AB} = \mathbf{BA}$ always holds.

   (b) If the characteristic polynomial of a 3×3 matrix is $\lambda^3 - 3\lambda^2 - \lambda + 3$, the eigenvalues of the matrix are $1, 3, -3$.

   (c) Let $\{\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_n\}$ be a basis for a vector space $\mathbb{V}$. Then for any other $\vec{\mathbf{y}} \in \mathbb{V}$, $\{\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_n, \vec{\mathbf{y}}\}$ is also a basis for $\mathbb{V}$.

   (d) For any $m \times n$ matrix $\mathbf{A}$, $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ is consistent if and only if $\vec{\mathbf{b}}$ is in the column space of $\mathbf{A}$, that is, $\vec{\mathbf{b}} \in \text{Col}\,\mathbf{A}$

   (e) If $\mathbf{A}$ and $\mathbf{B}$ are nonsingular matrices, then $\left|(\mathbf{AB})^{-1}\right| = (|\mathbf{A}||\mathbf{B}|)^{-1}$

SOLUTION:

(a) **TRUE** Although not a general proof, the pattern for the 3×3 case is illustrative.

$$\mathbf{AB} = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} \begin{bmatrix} b_{11} & 0 & 0 \\ 0 & b_{22} & 0 \\ 0 & 0 & b_{33} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} & 0 & 0 \\ 0 & a_{22}b_{22} & 0 \\ 0 & 0 & a_{33}b_{33} \end{bmatrix}$$

$$= \begin{bmatrix} b_{11}a_{11} & 0 & 0 \\ 0 & b_{22}a_{22} & 0 \\ 0 & 0 & b_{33}a_{33} \end{bmatrix} = \begin{bmatrix} b_{11} & 0 & 0 \\ 0 & b_{22} & 0 \\ 0 & 0 & b_{33} \end{bmatrix} \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} = \mathbf{BA}$$

(b) **FALSE** $\lambda^3 - 3\lambda^2 - \lambda + 3 = \lambda^2(\lambda - 3) - (\lambda - 3) = \left(\lambda^2 - 1\right)(\lambda - 3) = (\lambda - 1)(\lambda + 1)(\lambda - 3) \implies \lambda = -1, 1, 3$ are eigenvalues.

(c) **FALSE** Adding another vector to a basis automatically makes the new set linearly dependent and therefore it cannot be a basis.

(d) **TRUE** The 3×2 system will demonstrate the equivalence. To be consistent, an $\vec{\mathbf{x}}$ must exist such that $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$. That is,

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

This can be written as

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

showing that $\vec{\mathbf{b}}$ is in the column space of $\mathbf{A}$.

(e) **TRUE**

$$\left|(\mathbf{AB})^{-1}\right| = \left|\mathbf{B}^{-1}\mathbf{A}^{-1}\right| = \left|\mathbf{B}^{-1}\right|\left|\mathbf{A}^{-1}\right| = \left(\frac{1}{|\mathbf{B}|}\right)\left(\frac{1}{|\mathbf{A}|}\right) = \frac{1}{|\mathbf{A}||\mathbf{B}|} = (|\mathbf{A}||\mathbf{B}|)^{-1}$$

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2. [2360/031225 (23 pts)] Let $\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$.

(a) (4 pts) Find the eigenvalues of $\mathbf{A}$ and their algebraic multiplicity.

(b) (7 pts) Find the geometric multiplicity of the eigenvalue whose algebraic multiplicity from part (a) is one. Find a basis for the eigenspace of this eigenvalue.

(c) (8 pts) Find a basis for the solution space of $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{0}}$. What is the dimension of the solution space?

(d) (4 pts) Is $\mathbf{A}^{\mathrm{T}}$ invertible? Explain briefly without doing any calculations.

SOLUTION:

(a) Since this is a lower triangular matrix, the eigenvalues lie along the diagonal. They are $\lambda = 0$ with algebraic multiplicity 3 and $\lambda = 1$ with algebraic multiplicity 1.

(b) We need to solve $(\mathbf{A} - \mathbf{I})\vec{\mathbf{x}} = \vec{\mathbf{0}}$.

$$\left[\begin{array}{rrrr|r} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rrrr|r} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

A basis for the eigenspace corresponding to $\lambda = 1$ is $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$. The geometric multiplicity of $\lambda = 1$ is 1.

(c)

$$\left[\begin{array}{rrrr|r} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rrrr|r} 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \implies \begin{aligned} x_1 &= -s - t \\ x_2 &= r \\ x_3 &= s \\ x_4 &= t \end{aligned}$$

A basis for the solution space is $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$. Its dimension is 3.

(d) No. Since $\mathbf{A}$ contains a row of zeros, its determinant vanishes and therefore the determinant of $\left|\mathbf{A}^{\mathrm{T}}\right| = |\mathbf{A}|$ also vanishes, implying that $\mathbf{A}^{\mathrm{T}}$ is not invertible.

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3. [2360/031225 (20 pts)] Consider the linear system
$$\begin{aligned} x_2 + 2x_3 &= 1 \\ 3x_1 + 15x_2 + 6x_3 &= 9, \\ 6x_1 + 29x_2 + 10x_3 &= k \end{aligned}$$
where k is a real constant.

There is a single value of k that makes the system consistent. Find that value and then solve the system using that value for k by finding the RREF of an appropriate matrix. Use the Nonhomogeneous Principle to write the general solution in the form $\vec{\mathbf{x}} = \vec{\mathbf{x}}_h + \vec{\mathbf{x}}_p$, clearly indicating $\vec{\mathbf{x}}_h$ and $\vec{\mathbf{x}}_p$.

SOLUTION:

$$\left[\begin{array}{rrr|r} 0 & 1 & 2 & 1 \\ 3 & 15 & 6 & 9 \\ 6 & 29 & 10 & k \end{array}\right] \rightarrow \left[\begin{array}{rrr|c} 0 & 1 & 2 & 1 \\ 1 & 5 & 2 & 3 \\ 0 & -1 & -2 & k-18 \end{array}\right] \rightarrow \left[\begin{array}{rrr|c} 1 & 5 & 2 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & k-17 \end{array}\right] \rightarrow \left[\begin{array}{rrr|c} 1 & 0 & -8 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & k-17 \end{array}\right] \implies k = 17 \text{ to be consistent}$$

With $k = 17$, we have

$$\left[\begin{array}{ccc|c} 1 & 0 & -8 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 = -2 + 8t \\ x_2 = 1 - 2t \\ x_3 = t \end{array} \implies \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{t\begin{bmatrix} 8 \\ -2 \\ 1 \end{bmatrix}}_{\vec{\mathbf{x}}_h} + \underbrace{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}}_{\vec{\mathbf{x}}_p}, \quad t \in \mathbb{R}$$

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4. [2360/031225 (16 pts)] Consider the following vectors in $\mathbb{R}^3$:

$$\vec{\mathbf{x}} = \begin{bmatrix} 14 \\ 3 \\ -19 \end{bmatrix}, \vec{\mathbf{u}} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \vec{\mathbf{v}} = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}, \vec{\mathbf{w}} = \begin{bmatrix} 11 \\ 2 \\ -15 \end{bmatrix}$$

Show all your work to justify your answers to the following questions.

(a) (8 pts) Is $\vec{\mathbf{x}} \in \text{span}\ \{\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}}\}$?

(b) (8 pts) Is $\text{span}\ \{\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}}\} = \mathbb{R}^3$?

SOLUTION:

(a) Can c_1, c_2, c_3 be found such that $c_1\vec{\mathbf{u}} + c_2\vec{\mathbf{v}} + c_3\vec{\mathbf{w}} = \vec{\mathbf{x}}$?

$$c_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} 11 \\ 2 \\ -15 \end{bmatrix} = \begin{bmatrix} 14 \\ 3 \\ -19 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 11 & 14 \\ 2 & 1 & 2 & 3 \\ -1 & 3 & -15 & -19 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -2 & 11 & 14 \\ 0 & 5 & -20 & -25 \\ 0 & 1 & -4 & -5 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 3 & 4 \\ 0 & 1 & -4 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} c_1 = 4 - 3t \\ c_2 = -5 + 4t \\ c_3 = t \end{array}$$

From this we can conclude that $\vec{\mathbf{x}} = 4\vec{\mathbf{u}} - 5\vec{\mathbf{v}} + 0\vec{\mathbf{w}}$ showing that $\vec{\mathbf{x}} \in \text{span}\ \{\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}}\}$. Note: there are other choices for c_1, c_2, c_3 based on other values of t.

(b) Since the set contains 3 vectors in a vector space of dimension 3, this is equivalent to asking if $\{\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}}\}$ is a basis for $\mathbb{R}^3$. This will be the case if the vectors are linearly independent. This, in turn, is equivalent to showing that the only solution to $c_1\vec{\mathbf{u}} + c_2\vec{\mathbf{v}} + c_3\vec{\mathbf{w}} = \vec{\mathbf{0}}$ is $c_1 = c_2 = c_3 = 0$. Furthermore, this can be done by deciding if the trivial solution is the unique solution of

$$\begin{bmatrix} 1 & -2 & 11 \\ 2 & 1 & 2 \\ -1 & 3 & -15 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{vmatrix} 1 & -2 & 11 \\ 2 & 1 & 2 \\ -1 & 3 & -15 \end{vmatrix} = 11(-1)^{1+3} \begin{vmatrix} 2 & 1 \\ -1 & 3 \end{vmatrix} + 2(-1)^{2+3} \begin{vmatrix} 1 & -2 \\ -1 & 3 \end{vmatrix} - 15(-1)^{3+3} \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} = 11(7) - 2(1) - 15(5) = 0$$

This shows that nontrivial solutions exist, implying that the vectors are linearly dependent and therefore cannot be a basis for $\mathbb{R}^3$, implying that $\text{span}\ \{\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}}\} \neq \mathbb{R}^3$. Note: the RREF from part (a) with the last column containing all zeroes, is also justification for showing that the vectors are linearly dependent and thus cannot form a basis.

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5. [2360/031225 (16 pts)] Consider the set $\left\{t^2 + 3t - \frac{5}{4}, -2t^2 + t - 1, \frac{1}{2} - t\right\}$.

(a) (8 pts) Show that the Wronskian cannot be used to decide whether or not the set is linearly independent.

(b) (8 pts) Does the set form a basis for $\mathbb{P}_2$? Justify your answer.

SOLUTION:

(a)

$$W(t) = \begin{vmatrix} t^2+3t-\frac{5}{4} & -2t^2+t-1 & -t+\frac{1}{2} \\ 2t+3 & -4t+1 & -1 \\ 2 & -4 & 0 \end{vmatrix}$$

$$= \left(-t+\frac{1}{2}\right)(-1)^{1+3}\begin{vmatrix} 2t+3 & -4t+1 \\ 2 & -4 \end{vmatrix} - 1(-1)^{2+3}\begin{vmatrix} t^2+3t-\frac{5}{4} & -2t^2+t-1 \\ 2 & -4 \end{vmatrix}$$

$$= \left(-t+\frac{1}{2}\right)(-8t-12+8t-2) + \left(-4t^2-12t+5+4t^2-2t+2\right)$$

$$= \left(-t+\frac{1}{2}\right)(-14) + (-14t+7)$$

$$= 14t - 7 - 14t + 7 = 0$$

Since the Wronskian vanishes, this tells us nothing about the linear dependence of the vectors in the set.

(b) There are three vectors in the set and the dimension of $\mathbb{P}_2$ is 3. If the vectors are linearly independent, they will form a basis, otherwise they will not. We need to determine if the only solution to the following is $c_1 = c_2 = c_3 = 0$.

$$c_1\left(t^2+3t-\frac{5}{4}\right) + c_2\left(-2t^2+t-1\right) + c_3\left(-t+\frac{1}{2}\right) = 0 + 0t + 0t^2$$

$$c_1 - 2c_2 = 0$$

$$3c_1 + c_2 - c_3 = 0$$

$$-\frac{5}{4}c_1 - c_2 + \frac{1}{2}c_3 = 0$$

$$\begin{bmatrix} 1 & -2 & 0 \\ 3 & 1 & -1 \\ -\frac{5}{4} & -1 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{vmatrix} 1 & -2 & 0 \\ 3 & 1 & -1 \\ -\frac{5}{4} & -1 & \frac{1}{2} \end{vmatrix} = 1(-1)^{1+1}\begin{vmatrix} 1 & -1 \\ -1 & \frac{1}{2} \end{vmatrix} - 2(-1)^{1+2}\begin{vmatrix} 3 & -1 \\ -\frac{5}{4} & \frac{1}{2} \end{vmatrix} = \left(\frac{1}{2}-1\right) + 2\left(\frac{3}{2}-\frac{5}{4}\right) = -\frac{1}{2} + 3 - \frac{5}{2} = 3 - \frac{6}{2} = 0$$

This indicates that the system has nontrivial solutions, implying that the vectors are linearly dependent. Therefore, they cannot form a basis for $\mathbb{P}_2$.

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6. [2360/031225 (15 pts)] For each of the following, determine if the given subset, $\mathbb{W}$, is a subspace of the given vector space, $\mathbb{V}$. Assume that standard operations apply in each case. Provide justification for your answers.

(a) (5 pts) $\mathbb{V} = \mathbb{R}^3$; $\mathbb{W}$ is the set of vectors of the form $[n \quad n \quad 2n]^{\mathrm{T}}$ where n is an integer.

(b) (5 pts) $\mathbb{V} = C(-\infty, \infty)$ (the set of functions that are continuous for all real numbers); $\mathbb{W}$ equals the set of all constant functions.

(c) (5 pts) $\mathbb{V} = \mathbb{M}_{nn}$; $\mathbb{W}$ is the set of $n \times n$ matrices with trace equal to -1.

SOLUTION:

(a) Not a subspace. Not closed under scalar multiplication. For example $\vec{\mathbf{u}} = [1 \quad 1 \quad 2]^{\mathrm{T}} \in \mathbb{W}$ but $\sqrt{2}\vec{\mathbf{u}} = \left[\sqrt{2} \quad \sqrt{2} \quad 2\sqrt{2}\right]^{\mathrm{T}} \notin \mathbb{W}$. Note that the set is closed under vector addition.

(b) Subspace. Let $\vec{\mathbf{u}} = f(x) = a$ and $\vec{\mathbf{v}} = g(x) = b$ be in $\mathbb{W}$ and let $p, q \in \mathbb{R}$. Then

$$p\vec{\mathbf{u}} + q\vec{\mathbf{v}} = (pf)(x) + (qg)(x) = pf(x) + qg(x) = pa + qb \in \mathbb{W}$$

(c) Not a subspace. The zero vector, $\vec{\mathbf{0}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, is not in $\mathbb{W}$. The set is also not closed under either vector addition or scalar multiplication. To see this, note that vectors in the set have the form $\begin{bmatrix} a & b \\ c & -(a+1) \end{bmatrix}$, where $a, b, c \in \mathbb{R}$.

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