1. The following are unrelated:

   (a) (8 pts) Suppose $y$ is a function of $x$, find $y'$ if $\sin(xy) = \dfrac{2}{3}$.

   (b) (8 pts) Suppose $f(x) = \sqrt[3]{-x + |x|}$. Find the derivative of $f(x)$ for all $x < 0$.

   (c) (8 pts) Find $\dfrac{d^2y}{dx^2}$ at $x = \dfrac{\pi}{4}$ if $\dfrac{dy}{dx} = x\sec^2(x) + \tan(x)$.

   (d) (8 pts) For a differentiable function $g(x)$, find $\dfrac{d}{dx}\left(\dfrac{g\left(x^2\right)}{x}\right)$.

**Solution:**

(a) Apply implicit differentiation, the chain rule, and the product rule.

$$\frac{d}{dx}\left[\sin(xy)\right] = \frac{d}{dx}\left[2/3\right]$$

$$\cos(xy) \cdot \frac{d}{dx}\left[xy\right] = 0$$

$$\cos(xy) \cdot \big(x \cdot y' + y \cdot 1\big) = 0$$

Since $\cos(xy) = \sqrt{1 - \sin^2(xy)}$ and $\sin(xy) = 2/3$, the value of $\cos(xy)$ can not equal zero. That leads to the following:

$$x \cdot y' + y = 0$$

$$y' = \boxed{-y/x}$$

(b)

$$|x| = \begin{cases} x & , \quad x \geq 0 \\ -x & , \quad x < 0 \end{cases}$$

In this problem, only negative values of $x$ are considered, so that $|x| = -x$.

$$\begin{aligned} f(x) &= \sqrt[3]{-x + |x|} \\ &= \sqrt[3]{-x - x} \\ &= (-2x)^{1/3} \end{aligned}$$

$$
\begin{aligned}
f'(x) &= \frac{1}{3}(-2x)^{-2/3} \cdot \frac{d}{dx}[-2x] \\
&= \frac{1}{3} \cdot (-1)^{-2/3} \cdot 2^{-2/3} \cdot x^{-2/3} \cdot (-2) \\
&= -\frac{1}{3} \cdot 2^{1/3} \cdot x^{-2/3} \\
&= \boxed{-\frac{1}{3} \cdot \sqrt[3]{\frac{2}{x^2}}}
\end{aligned}
$$

(c)

$$
\frac{dy}{dx} = x \sec^2 x + \tan x
$$

$$
\begin{aligned}
\frac{d^2y}{dx^2} &= \left(x \cdot \frac{d}{dx}[\sec^2 x] + \sec^2 x \cdot 1\right) + \sec^2 x \\
&= x \cdot (2 \sec x) \cdot \frac{d}{dx}[\sec x] + 2 \sec^2 x \\
&= 2x \sec x \cdot (\sec x \tan x) + 2 \sec^2 x \\
&= 2 \sec^2 x (x \tan x + 1)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
y''(\pi/4) &= 2 \sec^2(\pi/4)(\pi/4 \cdot \tan(\pi/4) + 1) \\
&= 2 \cdot (\sqrt{2})^2 \cdot (\pi/4 \cdot 1 + 1) \\
&= \boxed{\pi + 4}
\end{aligned}
$$

(d) Begin with the quotient rule.

$$\begin{aligned}
\frac{d}{dx}\left[\frac{g\left(x^2\right)}{x}\right] &= \frac{x \cdot \frac{d}{dx}[g\left(x^2\right)] - g\left(x^2\right) \cdot 1}{x^2} \\
&= \frac{x \cdot g'\left(x^2\right) \cdot \frac{d}{dx}[x^2] - g\left(x^2\right)}{x^2} \\
&= \frac{x \cdot g'\left(x^2\right) \cdot 2x - g\left(x^2\right)}{x^2} \\
&= \frac{2x^2\, g'\left(x^2\right) - g\left(x^2\right)}{x^2} \\
&= \boxed{2\, g'\left(x^2\right) - \frac{g\left(x^2\right)}{x^2}}
\end{aligned}$$

2. (26 pts) Consider the function $y = x\sqrt{2+4x}$, with domain $\left[-\frac{1}{2}, \infty\right)$, to answer the following.

(a) Find the $x$ and $y$-intercepts of the function.

(b) The first derivative is $y' = \dfrac{2+6x}{\sqrt{2+4x}}$. On what intervals is $y$ increasing? Decreasing?

(c) Find $x$ and $y$ coordinates of the local maximum and minimum extrema, if any.

(d) Find the absolute maximum and absolute minimum values of $y$ on the interval $\left[-\frac{1}{2}, 3\right]$.

**Solution:**

(a) $y(0) = 0$ implies that the location of the $y$-intercept is $\boxed{(0,0)}$.

The $x$-intercepts must be located at values of $x$ for which $x\sqrt{2+4x} = 0$. $x = 0$ and $x = -1/2$ are the two values that satisfy the requirement. So, the $x$-intercepts are located at $\boxed{(0,0)}$ and $\boxed{(-1/2,0)}$.

(b) We're given that $y' = \dfrac{2+6x}{\sqrt{2+4x}}$.

The denominator of $y'$ is positive on $(-1/2, \infty)$.

The numerator of $y'$ is negative for $x < -1/3$ and it is positive for $x > -1/3$.

Therefore, $y' < 0$ on $(-1/2, -1/3)$ and $y' > 0$ on $(-1/3, \infty)$, so that $y$ is decreasing and increasing on those intervals, respectively.

Finally, although $y'$ is undefined at $x = -1/2$, we can see that $y(-1/2) > y(x)$ for all $x$ values in $(-1/2, -1/3)$, so that by the definition of a decreasing function, $y$ is decreasing on the entire interval $[-1/2, -1/3)$, including the left endpoint.

So, in summary:

$$y \text{ is increasing on } \boxed{(-1/3, \infty)}$$

$$y \text{ is decreasing on } \boxed{[-1/2, -1/3)}$$

(c) The critical numbers of $y$ are all $x$ values in the domain of $y$ such that either $y'(x) = 0$ or $y'(x)$ is undefined. From the derivative expression in part (c), we see that $y'(-1/2)$ is undefined and $y'(-1/3) = 0$. Since $x = -1/2$ and $x = -1/3$ are both in the domain of $y$, they are the critical numbers of $y$, so that those two values of $x$ are the only possible locations of local extrema of $y$.

Since a local extremum can not occur on the boundary of an interval, there is no local extremum at $x = -1/2$.

Since $y$ is continuous at $x = -1/3$ and $y'$ transitions from negative to positive at that location, the First Derivative Test indicates that $y$ has a local minimum at $x = -1/3$.

$$y(-1/3) = -\frac{1}{3}\sqrt{2 + 4 \cdot \left(-\frac{1}{3}\right)} = -\frac{1}{3}\sqrt{\frac{2}{3}}$$

Therefore, $y$ has a $\boxed{\text{local minimum at } \left(-\frac{1}{3}, -\frac{\sqrt{2}}{3\sqrt{3}}\right)}$

(d) Since $y$ is continuous on the closed interval $[-1/2, 3]$, the Extreme Value Theorem indicates that $y$ attains absolute maximum and minimum values on that interval, and the Closed Interval Method can be used to identify them. Specifically, evaluate $y$ at the boundaries of the domain and at the critical number in the interior of the domain.

$$y\left(-\frac{1}{2}\right) = 0 \qquad \text{(from part (a))}$$

$$y\left(-\frac{1}{3}\right) = -\frac{\sqrt{2}}{3\sqrt{3}} \qquad \text{(from part (c))}$$

$$y(3) = 3\sqrt{2 + 12} = 3\sqrt{14}$$

Upon identifying the maximum and minimum of the preceding three values, we conclude the following for the interval $[-1/2, 3]$:

$$y \text{ attains a maximum value of } \boxed{3\sqrt{14}}$$

$$y \text{ attains a minimum value of } \boxed{-\frac{\sqrt{2}}{3\sqrt{3}}}$$

3. (12 pts) For two resistors, $R_1$ and $R_2$, connected in parallel, the combined electrical resistance, $R$, is given by $(R)^{-1} = (R_1)^{-1} + (R_2)^{-1}$ where $R$, $R_1$, and $R_2$ are all functions of time and are measured in ohms. Suppose $R_1$ and $R_2$ are each increasing at a rate of $\frac{1}{2}$ ohms per second. At what rate is the combined resistance, $R$, changing when $R_1$ = 2 ohms and $R_2$ = 4 ohms?

**Solution:**

Since the goal is to determine the value of $dR/dt$ under the given conditions, we begin by differentiating with respect to $t$.

$$\frac{d}{dt}\left[R^{-1}\right] = \frac{d}{dt}\left[{R_1}^{-1} + {R_2}^{-1}\right]$$

$$-R^{-2} \cdot \frac{dR}{dt} = -{R_1}^{-2} \cdot \frac{dR_1}{dt} - {R_2}^{-2} \cdot \frac{dR_2}{dt}$$

$$\frac{1}{R^2} \cdot \frac{dR}{dt} = \frac{1}{{R_1}^2} \cdot \frac{dR_1}{dt} + \frac{1}{{R_2}^2} \cdot \frac{dR_2}{dt}$$

When $R_1 = 2$ and $R_2 = 4$, the value of the combined resistance can be determined as follows:

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

$$\frac{1}{R} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$R = \frac{4}{3}$$

The problem statement indicates that the value of both $dR_1/dt$ and $dR_2/dt$ equals $1/2$. Therefore,

$$\frac{1}{R^2} \cdot \frac{dR}{dt} = \frac{1}{{R_1}^2} \cdot \frac{dR_1}{dt} + \frac{1}{{R_2}^2} \cdot \frac{dR_2}{dt}$$

$$\frac{1}{(4/3)^2} \cdot \frac{dR}{dt} = \frac{1}{2^2} \cdot \frac{1}{2} + \frac{1}{4^2} \cdot \frac{1}{2}$$

$$\frac{dR}{dt} = \left(\frac{4}{3}\right)^2 \left(\frac{1}{8} + \frac{1}{32}\right)$$

$$\frac{dR}{dt} = \frac{16}{9} \cdot \frac{5}{32} = \boxed{\frac{5}{18} \text{ ohms per second}}$$

4. (8 pts) A company, Better Boulder Dice (BBD), is going to produce new metallic dice in the shape of a cube. Suppose $x$ represents the edge length of a metal cube.

(a) The volume of a cube is $V(x) = x^3$. Find $dV$, the differential of $V$.

(b) The edges of each cube are designed to have a length of 2 cm, but the machine creating the cube produces edge lengths of $2.01$ cm. Use differentials to estimate $\Delta V$, the difference between the designed volume and the machine-produced volume.

**Solution:**

(a)

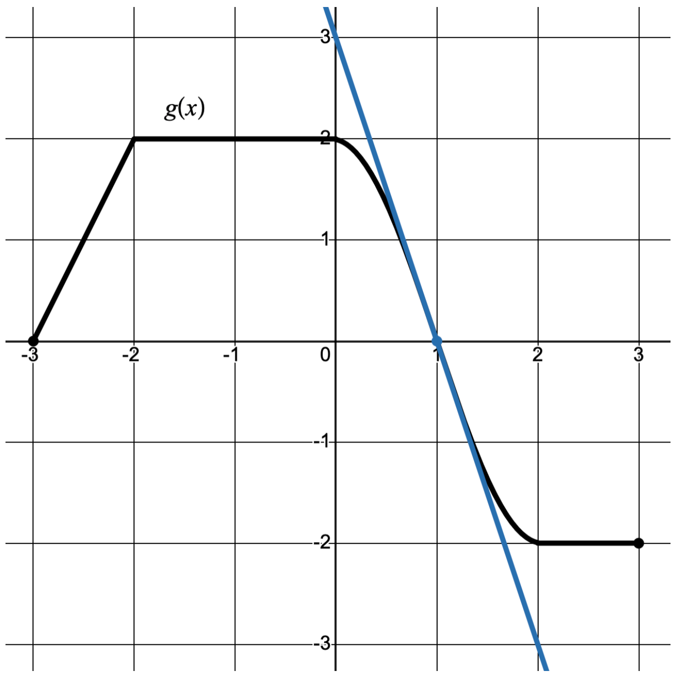
$$\frac{dV}{dx} = 3x^2$$

$$dV = \boxed{3x^2\,dx}$$

(b) The value of $x$ is 2 and the value of $dx$ is $2.01 - 2 = 0.01$. Therefore,

$$\Delta V \approx dV = 3 \cdot 2^2 \cdot 0.01 = \boxed{0.12 \text{ cm}^3}$$

5. (22 points)




Shown above is the graph of $y = g(x)$ and the tangent line to $g$ at $(1, 0)$. The function is differentiable on $(-3, -2) \cup (-2, 3)$.

(a) Sketch the graph of $y = g'(x)$. Clearly label the tick marks.

(b) Use the linearization of $g$ at $a = 1$ to estimate the value of $g(1.3)$.

(c) The mean value theorem states that there exists a value of $c$ in $(-2, 3)$ such that $g'(c)$ equals a certain value.

i. What is the value of $g'(c)$?

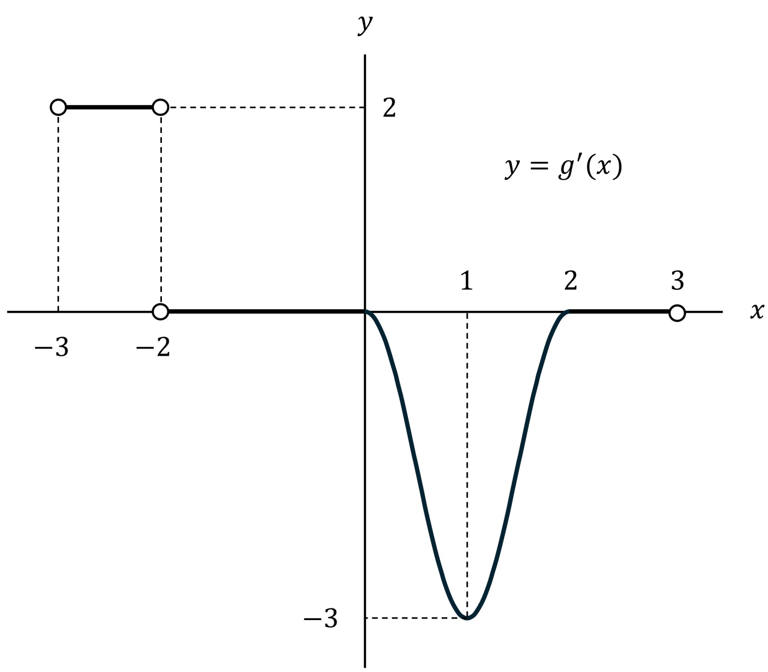
ii. Suppose we wish to narrow down the possible values for $c$. In which of the following six intervals can $c$ be found? Circle all possible answers. No explanation is necessary.

$(-3, -2)$ $\qquad$ $(-2, -1)$ $\qquad$ $(-1, 0)$

$(0, 1)$ $\qquad$ $(1, 2)$ $\qquad$ $(2, 3)$

**Solution:**

(a)




Note that $g'(1)$ is the slope of the tangent line in the $y = g(x)$ graph, which is

$$g'(1) = \frac{\Delta y}{\Delta x} = \frac{0-3}{1-0} = -3$$

(b)

$$g(x) \approx L(x) = g(1) + g'(1)(x-1) \qquad \text{“near” } x = 1$$

The point of tangency in the $y = g(x)$ graph is $(1, 0)$. Therefore, $g(1) = 0$.

The value $g'(1) = -3$ was determined in part (a).

It follows that the linearization of $g(x)$ “near” $x = 1$ is given by

$$g(x) \approx L(x) = 0 - 3(x-1) = -3(x-1)$$

Therefore,

$$g(1.3) \approx L(1.3) = -3(1.3-1) = \boxed{-0.9}$$

(c) i. Since $g(x)$ is continuous on $[-2, 3]$ and is differentiable on $(-2, 3)$, the Mean Value Theorem can be applied to $g(x)$ on the interval $[-2, 3]$ to state hat there exists some number $c$ in the interval $(-2, 3)$ such that

$$g'(c) = \frac{g(3) - g(-2)}{3 - (-2)} = \frac{-2 - 2}{5} = \boxed{-\frac{4}{5}}$$

ii. The graph of $y = g'(x)$ that was constructed in part (a) indicates that $g'(x)$ can only attain a value of $-4/5$ on the intervals $\boxed{(0, 1)}$ and $\boxed{(1, 2)}$