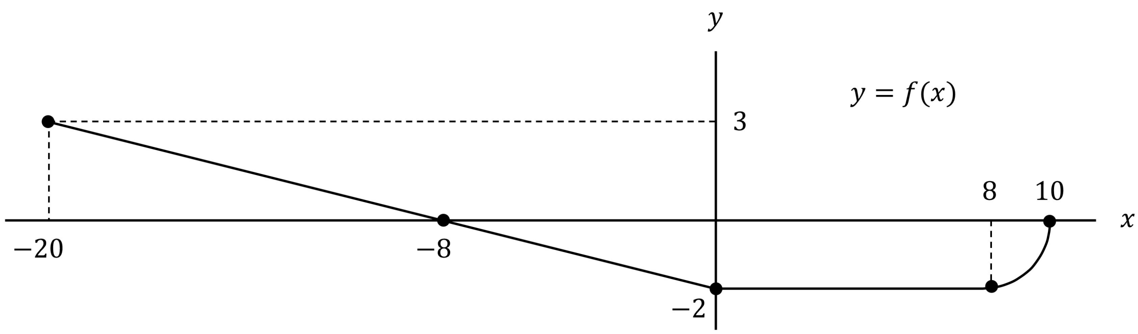
1. (15 pts) Parts (a) and (b) are not related.

   (a) The graph of $y = f(x)$, which consists of two line segments and a quarter circle, is depicted below on the interval $[-20, 10]$. Determine the average value of $f(x)$ on the interval $[-20, 10]$.



Solution:

$$f_{ave} = \frac{1}{10-(-20)}\int_{-20}^{10} f(x)\,dx = \frac{1}{30}\int_{-20}^{10} f(x)\,dx$$

$\displaystyle\int_{-20}^{10} f(x)\,dx$ represents the net area of the region between the curve $y = f(x)$ and the x-axis. That region consists of two triangular sub-regions, a rectangular sub-region, and a sub-region that is a sector of a circle. The left-most triangular region lies above the x-axis, so its area contributes a positive value toward the overall net area. The other three regions all lie below the x-axis, so their areas contribute negative values toward the overall net area. The areas of the various sub-regions are as follows:

$$\text{area of left-most triangle} = \frac{1}{2}(-8-(-20))(3-0) = \frac{1}{2}(12)(3) = 18$$

$$\text{area of right-most triangle} = \frac{1}{2}(0-(-8))(0-(-2)) = \frac{1}{2}(8)(2) = 8$$

$$\text{area of rectangle} = (8-0)(0-(-2)) = (8)(2) = 16$$

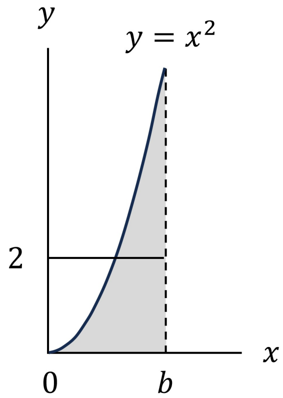
$$\text{area of sector of circle} = \frac{1}{4}\left(\pi(10-8)^2\right) = \frac{1}{4}(4\pi) = \pi$$

Therefore,

$$\int_{-20}^{10} f(x)\,dx = 18 - 8 - 16 - \pi = -6 - \pi$$

$$f_{ave} = \frac{1}{30}(-6-\pi) = \boxed{-\frac{\pi+6}{30}}$$

(b) For what value of $b > 0$ is the area of the shaded region in the following graph equal to the area of the rectangle whose corners are located at $(0, 0)$, $(b, 0)$, $(0, 2)$, and $(b, 2)$?



Solution:

Based on the description in the problem statement, 2 is the average value of $y = x^2$ on the interval $[0, b]$.

$$y_{ave} = \frac{1}{b-0}\int_0^b x^2\,dx = 2$$

$$\frac{1}{b}\cdot\frac{x^3}{3}\bigg|_0^b = 2$$

$$\frac{1}{b}\cdot\left(\frac{b^3}{3} - \frac{0^3}{3}\right) = 2$$

$$\frac{b^2}{3} = 2$$

$$b^2 = 6$$

$$b = \boxed{\sqrt{6}}$$

2. (16 pts) Parts (a) and (b) are not related.

(a) Evaluate the following derivative. Do **not** simplify your answer.

$$\frac{d}{dx}\int_{x^5+1}^{\tan x}\frac{1}{t^4+3}\,dt$$

Solution:

Since $\dfrac{1}{t^4+3}$ is continuous on $(-\infty,\infty)$, the following relationship holds for any real number c and for any value of x for which $\tan x$ is defined:

$$\int_{x^5+1}^{\tan x}\frac{1}{t^4+3}\,dt=\int_{c}^{\tan x}\frac{1}{t^4+3}\,dt-\int_{c}^{x^5+1}\frac{1}{t^4+3}\,dt$$

It follows that

$$\frac{d}{dx}\int_{x^5+1}^{\tan x}\frac{1}{t^4+3}\,dt=\frac{d}{dx}\int_{c}^{\tan x}\frac{1}{t^4+3}\,dt-\frac{d}{dx}\int_{c}^{x^5+1}\frac{1}{t^4+3}\,dt$$

Therefore, Part 1 of the Fundamental Theorem of Calculus and the Chain Rule indicate that

$$\frac{d}{dx}\int_{x^5+1}^{\tan x}\frac{1}{t^4+3}\,dt=\frac{1}{\tan^4 x+3}\cdot\frac{d}{dx}[\tan x]-\frac{1}{(x^5+1)^4+3}\cdot\frac{d}{dx}[x^5+1]$$

$$=\boxed{\frac{\sec^2 x}{\tan^4 x+3}-\frac{5x^4}{(x^5+1)^4+3}}$$

(b) For the following function $g(x)$, find the value of $g'(\pi/2)$. Fully simplify your final answer.

$$g(x) = \int_{\pi/2}^{5x-\pi/2} \sqrt{\cos t + 8}\, dt$$

Solution: Part 1 of the Fundamental Theorem of Calculus and the Chain Rule indicate that

$$\begin{aligned} g'(x) = \frac{d}{dx} \int_{\pi/2}^{5x-\pi/2} \sqrt{\cos t + 8}\, dt &= \sqrt{\cos(5x - \pi/2) + 8} \cdot \frac{d}{dx}[5x - \pi/2] \\ &= 5\sqrt{\cos(5x - \pi/2) + 8} \end{aligned}$$

Therefore,

$$\begin{aligned} g'(\pi/2) &= 5\sqrt{\cos(5 \cdot \pi/2 - \pi/2) + 8} \\ &= 5\sqrt{\cos(2\pi) + 8} \\ &= 5\sqrt{1 + 8} = \boxed{15} \end{aligned}$$

3. (34 pts) Parts (a), (b), and (b) are not related.

(a) Evaluate $\displaystyle\int \frac{\sin x}{(2\cos x - 1)^{5/6}}\,dx$.

Solution:

Apply u-substitution with $u = 2\cos x - 1$.

$$\frac{du}{dx} = -2\sin x$$

$$-\frac{1}{2}\cdot du = \sin x\,dx$$

$$\begin{aligned}\int \frac{\sin x}{(2\cos x - 1)^{5/6}}\,dx &= -\frac{1}{2}\int u^{-5/6}\,du \\ &= -\frac{1}{2}\cdot\frac{u^{1/6}}{1/6} + C \\ &= \boxed{-3(2\cos x - 1)^{1/6} + C}\end{aligned}$$

(b) Evaluate $\displaystyle\int (x+1)(x-1)^{20}\,dx$

Solution:

Apply u-substitution with $u = x - 1$. So, $du/dx = 1$, which implies that $du = dx$.

Also, $u = x - 1$ implies that $x + 1 = u + 2$.

$$\begin{aligned}\int (x+1)(x-1)^{20}\,dx &= \int (u+2)u^{20}\,du \\ &= \int (u^{21} + 2u^{20})\,du \\ &= \frac{u^{22}}{22} + 2\cdot\frac{u^{21}}{21} + C \\ &= \boxed{\frac{(x-1)^{22}}{22} + \frac{2(x-1)^{21}}{21} + C}\end{aligned}$$

(c) Let $v(t) = \dfrac{1+t^2}{\sqrt{2+3t+t^3}}$ meters per second represent the velocity function of a particle. Determine the distance traveled by the particle from $t = 0$ second to $t = 2$ seconds. Include the correct unit of measurement.

Solution:

Since the position function, $s(t)$, is an antiderivative of the velocity function, $v(t)$, and $v(t)$ is continuous on the interval $[0, 2]$, the Evaluation Theorem (Part 2 of the Fundamental Theorem of Calculus) indicates that

$$\int_0^2 v(t)\,dt = s(2) - s(0)$$

Since $v(t) \geq 0$ on $[0, 2]$, the distance, D, traveled from $t = 0$ to $t = 2$ equals the net change in position, $s(2) - s(0)$. Therefore,

$$D = \int_0^2 \frac{1+t^2}{\sqrt{2+3t+t^3}}\,dt$$

To evaluate the integral, apply u-substitution with $u = 2 + 3t + t^3$.

$$\frac{du}{dt} = 3 + 3t^2 = 3(1+t^2)$$

$$du = 3(1+t^2)\,dt$$

The limits of integration must be changed accordingly:

$$t = 0 \quad \Rightarrow \quad u = 2 + (3)(1) + 1^3 = 6$$

$$t = 2 \quad \Rightarrow \quad u = 2 + (3)(2) + 2^3 = 16$$

$$\begin{aligned} D = \int_0^2 \frac{1+t^2}{\sqrt{2+3t+t^3}}\,dt &= \frac{1}{3}\int_6^{16} u^{-1/2}\,du \\ &= \frac{1}{3}\cdot\frac{u^{1/2}}{1/2}\bigg|_6^{16} \\ &= \frac{2}{3}\left(\sqrt{16} - \sqrt{6}\right) \\ &= \boxed{\frac{2}{3}\left(4 - \sqrt{6}\right)\text{ m}} \end{aligned}$$

4. (35 pts) Parts (a), (b), and (c) are not related.

(a) Find the numerical value of the lower Riemann sum (**not** the left Riemann sum) for the function $p(x) = x^2 - 2x - 3$ on the interval $[-4, 8]$ using $n = 3$ equal subintervals.

Solution:

If the interval $[-4, 8]$ is divided into three equal subintervals, the width of each subinterval will be

$$\Delta x = \frac{8-(-4)}{3} = \frac{12}{3} = 4$$

So, the interval $[-4, 8]$ is divided into three subintervals: $[-4, 0]$, $[0, 4]$, and $[4, 8]$.

The derivative of $p(x)$ is $p'(x) = 2x - 2 = 2(x-1)$, which is negative on $[-4, 1]$ and is positive on $[1, 8]$. Therefore, $p(x)$ is decreasing on $[-4, 0]$ and is increasing on $[4, 8]$. On the subinterval $[0, 4]$, $p(x)$ attains an absolute minimum value at $x = 1$, per the First Derivative Test. So, the minimum values of $h(x)$ on each of the three subintervals are as follows:

$$\begin{aligned}
[-4, 0]: &\quad p(0) = 0^2 - 2 \cdot 0 - 3 = -3 \\
[0, 4]: &\quad p(1) = 1^2 - 2 \cdot 1 - 3 = -4 \\
[4, 8]: &\quad p(4) = 4^2 - 2 \cdot 4 - 3 = 5
\end{aligned}$$

Therefore, the lower Riemann sum equals $\Delta x(-3 - 4 + 5) = 4 \cdot (-2) = \boxed{-8}$

(b) Find an expression for the right-hand Riemann sum R_n for the function $h(x) = \sqrt{x}$ on the interval $[2, 16]$ using n equal subintervals.

Your final answer should be in the form of a summation. Do **not** simplify the expression or evaluate the limit of the summation.

Solution:

If the interval $[2, 16]$ is divided into n equal subintervals, the width of each subinterval will be

$$\Delta x = \frac{16-2}{n} = \frac{14}{n}$$

The right-hand endpoint of subinterval i is located at

$$x_i = 2 + i \cdot \Delta x = 2 + i \cdot \frac{14}{n}$$

Therefore,

$$R_n = \boxed{\sum_{i=1}^{n} \sqrt{2 + \frac{14i}{n}} \cdot \frac{14}{n}}$$

(c) Evaluate the following limit using summation formulas and fully simplify your final answer. Do not use L'Hôpital's Rule or a dominance of powers argument when evaluating the limit.

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{i^3}{n^3}\cdot\frac{1}{n}+3\cdot\frac{i}{n}\cdot\frac{1}{n}\right)$$

Solution:

$$\begin{aligned}
\sum_{i=1}^{n}\left(\frac{i^3}{n^3}\cdot\frac{1}{n}+3\cdot\frac{i}{n}\cdot\frac{1}{n}\right) &= \frac{1}{n^4}\sum_{i=1}^{n} i^3+\frac{3}{n^2}\sum_{i=1}^{n} i \\
&= \frac{1}{n^4}\left[\frac{n(n+1)}{2}\right]^2+\frac{3}{n^2}\left[\frac{n(n+1)}{2}\right] \\
&= \frac{n^2(n+1)^2}{2^2n^4}+\frac{3n(n+1)}{2n^2} \\
&= \frac{(n+1)^2}{4n^2}+\frac{3(n+1)}{2n}
\end{aligned}$$

$$\begin{aligned}
\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{i^3}{n^3}\cdot\frac{1}{n}+3\cdot\frac{i}{n}\cdot\frac{1}{n}\right) &= \lim_{n\to\infty}\frac{(n+1)^2}{4n^2}+\lim_{n\to\infty}\frac{3(n+1)}{2n} \\
&= \lim_{n\to\infty}\frac{n^2+2n+1}{4n^2}+\lim_{n\to\infty}\frac{3n+3}{2n} \\
&= \lim_{n\to\infty}\frac{n^2+2n+1}{4n^2}\cdot\frac{1/n^2}{1/n^2}+\lim_{n\to\infty}\frac{3n+3}{2n}\cdot\frac{1/n}{1/n} \\
&= \lim_{n\to\infty}\frac{1+2/n+1/n^2}{4}+\lim_{n\to\infty}\frac{3+3/n}{2} \\
&= \frac{1+0+0}{4}+\frac{3+0}{2} \\
&= \frac{1}{4}+\frac{3}{2}=\boxed{\frac{7}{4}}
\end{aligned}$$