1. [2360/021225 (10 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given. Please write your answers in a single column separate from any work you do to arrive at the answer.

   (a) All solutions of the differential equation $\frac{\mathrm{d}z}{\mathrm{d}x} = -1 - x^4 - z^4$ approach $-\infty$ as $x \to \infty$ regardless of the initial condition.

   (b) The equation $x' = (t-3)(x+2)^2(x-1)$ has three equilibrium solutions.

   (c) There exists a single real value of a that makes $a(\sin^2 t)y'' + \left(e^t - ay^{(4)}\right) y' - y + a = 0$ a linear, homogeneous, separable differential equation.

   (d) If L is a linear operator such that $L(\vec{\mathbf{u}}) = 2t$ and $L(\vec{\mathbf{w}}) = -t$, then $\vec{\mathbf{u}} + 2\vec{\mathbf{w}}$ is a solution to $L(\vec{\mathbf{y}}) = 0$.

   (e) $y^4 + 16y + x^4 - 8x^2 = 1$ is the implicit solution of $y' = \frac{4+y^3}{4x - x^3}$ passing through $(2, 1)$.

SOLUTION:

(a) **TRUE** The derivative of the solution is negative everywhere.

(b) **FALSE** The equation has two equilibrium solutions, $x = -2, 1$.

(c) **TRUE** If $a = 0$ we have $e^t y' - y = 0$.

(d) **TRUE** $L(\vec{\mathbf{u}} + 2\vec{\mathbf{w}}) = L(\vec{\mathbf{u}}) + 2L(\vec{\mathbf{w}}) = 2t + 2(-t) = 0$

(e) **FALSE** The initial condition gives $1^4 + 16(1) + 2^4 - 8(2)^2 = 1$ which is satisfied. However, implicit differentiation yields

$$4y^3y' + 16y' + 4x^3 - 16x = 0 \implies \left(4y^3 + 16\right) y' = 16x - 4x^3$$

$$y' = \frac{4\left(4x - x^3\right)}{4\left(y^3 + 4\right)} = \frac{4x - x^3}{y^3 + 4}$$

■

2. [2360/021225 (19 pts)] A simple model to describe paying off the debt on a credit card is given by $A' = 0.2A - 600$, $A(0) = A_0$ where $A(t)$ (t in years) is the amount of the debt (dollars).

   (a) (2 pts) What are the interest rate (%) and the yearly payment?

   (b) (3 pts) Find the equilibrium solution. Interpret, by writing a sentence or two, the meaning of the equilibrium solution with regard to paying off the loan.

   (c) (14 pts) If the initial amount of debt is \$2000, can the credit card debt be paid off in a finite amount of time? If so, find that time. If not, explain why not.

SOLUTION:

(a) The interest rate is 20% (yikes!) and yearly payment is \$600.

(b) The equilibrium solution is given by $0.2A - 600 = 0 \implies A = 3000$. If the amount of debt is \$3000, the interest gained on the debt exactly equals the payment, meaning that the debt on the card never changes (and never gets paid off).

(c) We'll use the integrating factor to solve the differential equation $A' - 0.2A = -600$.

$$p(t) = -0.2 \implies \int p(t)\,\mathrm{d}t = -0.2t \implies \mu(t) = e^{-0.2t}$$

$$\left(e^{-0.2t}A\right)' = -600e^{-0.2t}$$

$$\int \left(e^{-0.2t}A\right)'\,\mathrm{d}t = e^{-0.2t}A = -600\int e^{-0.2t}\,\mathrm{d}t = 3000e^{-0.2t} + C$$

$$A(t) = 3000 + Ce^{0.2t} \qquad \text{apply initial condition}$$

$$A(0) = 2000 = 3000 + C \implies C = -1000$$

$$A(t) = 3000 - 1000e^{0.2t} = 3000 - 1000e^{t/5}$$

The debt is paid off when $A(t) = 0$.

$$0 = 3000 - 1000e^{t/5} \implies \frac{3000}{1000} = e^{t/5} \implies t = 5\ln 3 \text{ years}$$

Yes, the loan can be paid off in a finite amount of time. Remark: Variation of parameters and separation of variables can also be used to find the general solution.

■

3. [2360/021225 (29 pts)] Consider the initial value problem $\left(t^2+4\right)\dfrac{\mathrm{d}y}{\mathrm{d}t} + 2ty = 4t,\ y(0) = 2$.

(a) (6 pts) Does Picard's Theorem guarantee that a unique solution to the IVP exists? Justify your answer.

(b) (8 pts) You are told to estimate $y(0.5)$ using 10 iterations of Euler's Method.

i. (2 pts) What is the step size, h?

ii. (3 pts) Find y_1, the first output of Euler's method using the step size you found in part (i).

iii. (3 pts) Will the estimate, y_1, from part (ii) be the same if you use a different step size? Explain briefly.

(c) (15 pts) Use variation of parameters (Euler-Lagrange Two Step Method) to solve the IVP.

SOLUTION:

(a) Rewrite the equation as $\dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{4t-2ty}{t^2+4}$. Then $f(t,y) = \dfrac{4t-2ty}{t^2+4}$ and $f_y(t,y) = -\dfrac{2t}{t^2+4}$ which are both continuous for all values of t and y and thus in any rectangle containing $(0,2)$. Therefore, Picard's Theorem guarantees a unique solution to the IVP.

(b) i. $h = \dfrac{1/2-0}{10} = \dfrac{1}{20}$

ii. $y(1/20) = 2 + \dfrac{1}{20}\left(\dfrac{4(0)-2(0)(2)}{0^2+4}\right) = 2.$

iii. Yes, since $f(0,2) = 0$, $y_1 = y_0 + hf(0,2) = y_0 + h(0) = y_0 = 2$, regardless of the value of h.

(c) Rewrite in standard form as $\dfrac{\mathrm{d}y}{\mathrm{d}t} + \dfrac{2ty}{t^2+4} = \dfrac{4t}{t^2+4}$. Solve the associated homogeneous problem using separation of variables:

$$\int \frac{\mathrm{d}y_h}{y_h} = -\int \frac{2t}{t^2+4}\,\mathrm{d}t \qquad \left(u = t^2+4\right)$$

$$\ln|y_h| = -\ln\left(t^2+4\right) + k$$

$$y_h = C\left(t^2+4\right)^{-1}$$

Set $y_p = v(t)\left(t^2+4\right)^{-1}$ and substitute into the nonhomogeneous equation:

$$-2tv(t)\left(t^2+4\right)^{-2} + v'(t)\left(t^2+4\right)^{-1} + \frac{2tv(t)\left(t^2+4\right)^{-1}}{\left(t^2+4\right)} = \frac{4t}{t^2+4}$$

$$v'(t) = 4t$$

$$\int v'(t)\,\mathrm{d}t = \int 4t\,\mathrm{d}t \implies v(t) = 2t^2 \implies y_p = \frac{2t^2}{t^2+4}$$

Apply the Nonhomogeneous Principle and the initial condition

$$y = y_h + y_p = \frac{C+2t^2}{t^2+4}$$

$$y(0) = 2 = \frac{C}{4} \implies C = 8$$

$$y(t) = \frac{2t^2+8}{t^2+4} = 2$$

■

4. [2360/021225 (12 pts)] A 1000 kiloliter (kL) holding pond at a wastewater treatment plant is half full. The water in the pond initially contains 7 grams of dissolved heavy metal contaminants. Water having $(2 + \cos t)$ grams/kL of the heavy metals flows into the pond from a mining operation at 3 kL/day. In addition, a mountain stream containing 4 grams/kL of the heavy metals empties into the pond at 2 kL/day. The well-mixed contaminated water exits the holding pond at 5 kL/day.

   (a) (9 pts) Write, but **DO NOT SOLVE**, an initial value problem that governs the amount, $c(t)$ (in grams), of the heavy metal contaminants in the pond at any time. Be sure to simplify your answer.

   (b) (3 pts) Provide the largest interval over which the solution to the IVP is valid. Again, do not solve the IVP but do provide a brief reason for your answer.

SOLUTION:

(a) The flow into the pond is 5 kL/day, the sum of the flow from the mine, 3 kL/day, and that from the stream 2 kL/day. The total flow out of the pond is 5 kL/day so that the volume of solution in the pond remains the same, 500 kL, for all time.

$$\frac{\mathrm{d}c}{\mathrm{d}t} = \left[(2+\cos t)\,\frac{\text{g}}{\text{kL}}\right]\left(3\,\frac{\text{kL}}{\text{day}}\right) + \left(4\,\frac{\text{g}}{\text{kL}}\right)\left(2\,\frac{\text{kL}}{\text{day}}\right) - \left(\frac{c(t)}{500}\,\frac{\text{g}}{\text{kL}}\right)\left(5\,\frac{\text{kL}}{\text{day}}\right) = 3(2+\cos t) + 8 - \frac{5c(t)}{500}$$

$$\frac{\mathrm{d}c}{\mathrm{d}t} + \frac{c}{100} = 3\cos t + 14,\ c(0) = 7$$

(b) Since the pond neither fills nor empties, the solution is valid for $0 \le t$.

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5. [2360/021225 (15 pts)] The population of Wookiees, w (in thousands of animals), on the planet Kashyyyk is governed by the differential equation $\mathrm{d}w/\mathrm{d}t = (10 - w)w - 25$, where t is measured in decades. Chewbacca leaves the planet when $t = 1$ at which time there are 4000 Wookiees alive $(w = 4)$. He returns several decades later to find out that the Wookiees went extinct while he was gone. Although brokenhearted, he wants to know the time, t_e, when the extinction occurred. Please find this time for him (and your grader).

SOLUTION:

$$\frac{\mathrm{d}w}{\mathrm{d}t} = 10w - w^2 - 25 = -(w^2 - 10w + 25) = -(w-5)^2$$

$$\int (w-5)^{-2}\,\mathrm{d}w = \int -\,\mathrm{d}t$$

$$-(w-5)^{-1} = -t + k$$

$$\frac{1}{w-5} = t + C \qquad \text{apply initial condition} \qquad \frac{1}{4-5} = 1 + C \implies C = -2$$

$$\frac{1}{w-5} = t - 2$$

The Wookiees are extinct when $w = 0$ giving $t_e = \dfrac{1}{0-5} + 2 = -\dfrac{1}{5} + 2 = \dfrac{9}{5}$ decades or 18 years. ■

6. [2360/021225 (15 pts)] No work needs to be shown and no partial credit will be given on this problem. The figure below shows the phase plane for the system

$$x' = x(4 - 2x - y)$$
$$y' = y(6 - 2x - 3y)$$

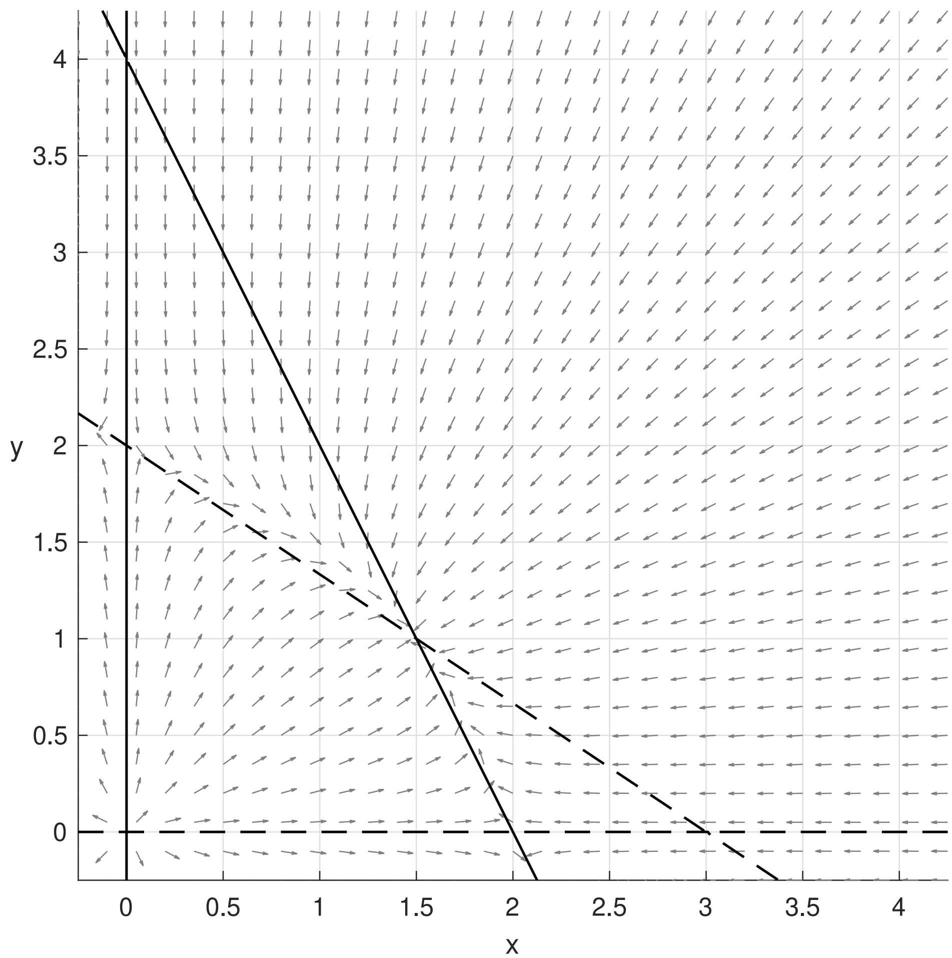
   (a) (2 pts) What do the solid lines represent?

   (b) (2 pts) What do the dashed lines represent?

   (c) (8 pts) Find all equilibrium solutions and determine their stability.

   (d) (3 pts) For each of the following initial conditions, find the limiting values (long term behavior) of x and y, that is, what are $\lim_{t\to\infty} x(t)$ and $\lim_{t\to\infty} y(t)$? Give your answers as an ordered pair (x, y).

      i. $(x(0), y(0)) = \left(\frac{1}{2}, 1\right)$ ii. $(x(0), y(0)) = (2, 0)$ iii. $(x(0), y(0)) = (3, 2)$



SOLUTION:

(a) v nullclines

(b) h nullclines

(c) intersection of the v and h nullclines - unstable: $(0,0)$, $(0,2)$, $(2,0)$; stable: $(1.5,1)$

(d) i. $(1.5,1)$ ii. $(2,0)$ iii. $(1.5,1)$

■