1. (37 pts) Evaluate the following integrals and simplify your answers.

   (a) $\displaystyle\int \frac{x}{x^2+4x+3}\,dx$

   (b) $\displaystyle\int \frac{\sqrt{x^2-9}}{x}\,dx$

   (c) $\displaystyle\int_0^1 \frac{dx}{x\,(1+(\ln x)^2)}$

   Solution:

   (a) Using partial fractions, we have

$$\frac{x}{x^2+4x+3} = \frac{x}{(x+3)(x+1)} = \frac{A}{x+3} + \frac{B}{x+1}.$$

   Solving

$$A(x+1) + B(x+3) = x$$

   gives

$$\begin{aligned} A + B &= 1 \\ A + 3B &= 0 \end{aligned}$$

   which has the solution $A = \frac{3}{2}$, $B = -\frac{1}{2}$. Thus

$$\int \frac{x}{x^2+4x+3}\,dx = \int \left(\frac{3}{2(x+3)} - \frac{1}{2(x+1)}\right) dx = \boxed{\frac{3}{2}\ln|x+3| - \frac{1}{2}\ln|x+1| + C}$$

   (b) Use a trig substitution with $x = 3\sec\theta$ and $dx = 3\sec\theta\tan\theta\,d\theta$. Then $\sqrt{x^2-9} = \sqrt{9\sec^2\theta - 9} = \sqrt{9\left(\sec^2\theta - 1\right)} = 3\tan\theta$. Therefore

$$\begin{aligned} \int \frac{\sqrt{x^2-9}}{x}\,dx &= \int \frac{3\tan\theta}{3\sec\theta}(3\sec\theta\tan\theta)\,d\theta \\ &= \int 3\tan^2\theta\,d\theta \\ &= \int 3\left(\sec^2\theta - 1\right) d\theta \\ &= 3\left(\tan\theta - \theta\right) + C \\ &= \boxed{\sqrt{x^2-9} - 3\sec^{-1}(x/3) + C}. \end{aligned}$$

(c) This is an improper integral.

$$\int_0^1 \frac{dx}{x\,(1+(\ln x)^2)} = \lim_{t\to 0^+} \int_t^1 \frac{dx}{x\,(1+(\ln x)^2)}$$

Let $u = \ln x$, $du = dx/x$.

$$\begin{aligned} &= \lim_{t\to 0^+} \int_{\ln t}^0 \frac{du}{1+u^2} \\ &= \lim_{t\to 0^+} \left[\tan^{-1} u\right]_{\ln t}^0 \\ &= \lim_{t\to 0^+} \left(\tan^{-1} 0 - \tan^{-1}(\ln t)\right) \\ &= 0 - (-\pi/2) = \boxed{\pi/2}. \end{aligned}$$

Note: $\lim\limits_{t\to 0^+} \ln t = -\infty$ and $\lim\limits_{u\to -\infty} \tan^{-1} u = -\pi/2$.

2. (10 pts) Determine whether $\displaystyle\int_{106}^{\infty} \frac{x^3}{\sqrt{x^{10}+\pi}}\,dx$ is convergent or divergent. Fully explain your reasoning.

Solution: For $x > 0$, we have

$$0 < \frac{x^3}{\sqrt{x^{10}+\pi}} < \frac{x^3}{\sqrt{x^{10}}} = \frac{x^3}{x^5} = \frac{1}{x^2}.$$

Because $\displaystyle\int_1^{\infty} \frac{1}{x^2}\,dx$ is a convergent p-integral ($p = 2 > 1$) and $\displaystyle\int_1^{106} \frac{1}{x^2}\,dx$ has finite area, the integral $\displaystyle\int_{106}^{\infty} \frac{1}{x^2}\,dx$ also is convergent.

Therefore the given integral is $\boxed{\text{convergent}}$ by the Comparison Theorem.

3. (38 pts) Consider the integral $I = \displaystyle\int_0^{\pi} x\cos(x/2)\,dx$.

(a) Estimate the value of I using the trapezoidal approximation T_3. Fully simplify your answer.

(b) Estimate the error for the approximation T_3. Express your answer in terms of π and simplify.

(c) Find the exact value of the integral $I = \displaystyle\int_0^{\pi} x\cos(x/2)\,dx$.

(d) Consider the region bounded by the curve $y = x\cos(x/2)$ and the x-axis on $[0, \pi]$. Suppose the region is rotated about the line $y = 2$ (which lies above the region). Set up (but do not evaluate) an integral to find the volume of the generated solid.

Solution:

(a) Let $f(x) = x\cos(x/2)$, $n = 3$, and $\Delta x = \frac{b-a}{n} = \frac{\pi}{3}$.

$$\begin{aligned} T_3 &= \frac{1}{2}(\Delta x)\left[f(0) + 2f(\pi/3) + 2f(2\pi/3) + f(\pi)\right] \\ &= \frac{1}{2}\cdot\frac{\pi}{3}\left[0 + 2\cdot\frac{\pi}{3}\cos(\pi/6) + 2\cdot\frac{2\pi}{3}\cos(\pi/3) + \pi\cos(\pi/2)\right] \end{aligned}$$

$$= \frac{\pi}{6}\left[0 + \frac{2\pi}{3}\cdot\frac{\sqrt{3}}{2} + \frac{4\pi}{3}\cdot\frac{1}{2} + 0\right]$$

$$= \frac{\pi}{6}\left(\frac{\sqrt{3}}{3}\pi + \frac{2}{3}\pi\right) = \boxed{\frac{\pi^2}{18}\left(\sqrt{3}+2\right)} = \boxed{\pi^2\left(\frac{\sqrt{3}}{18} + \frac{1}{9}\right)}$$

(b) First find an upper bound for $|f''|$.

$$\begin{aligned}
f'(x) &= -\frac{1}{2}x\sin(x/2) + \cos(x/2) \\
f''(x) &= -\frac{1}{4}x\cos(x/2) - \frac{1}{2}\sin(x/2) - \frac{1}{2}\sin(x/2) \\
&= -\frac{1}{4}x\cos(x/2) - \sin(x/2) \\
\left|f''(x)\right| &= \left|\frac{1}{4}x\cos(x/2) + \sin(x/2)\right| \\
&\le \frac{1}{4}|x|\,|\cos(x/2)| + |\sin(x/2)| \\
&\le \frac{1}{4}\pi + 1
\end{aligned}$$

On the interval $[0, \pi]$, $|x|$ attains a maximum value of π. Both $|\cos(x/2)|$ and $|\sin(x/2)|$ attain maximum values of 1, so $|f''| \le \pi/4 + 1$. Let $K = \pi/4 + 1$. Then an error estimate for T_3 is

$$|E_T| \le \frac{K(b-a)^3}{12n^2} = \frac{K\pi^3}{12\cdot 3^2} = \frac{K\pi^3}{108} \le \boxed{\frac{\pi^3(\pi/4+1)}{108}} = \boxed{\frac{\pi^4 + 4\pi^3}{432}}.$$

(c)

$$I = \int_0^{\pi} x\cos(x/2)\,dx$$

Use the Integration by Parts formula

$$\int u\,dv = uv - \int v\,du,$$

with

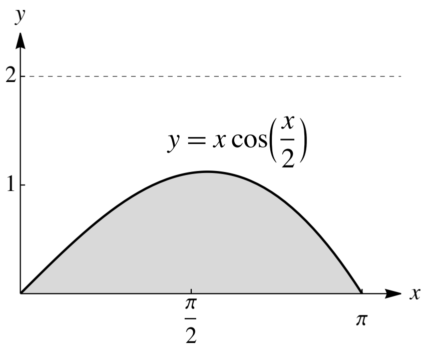
$$\begin{aligned}
u &= x & dv &= \cos(x/2)\,dx \\
du &= dx & v &= 2\sin(x/2).
\end{aligned}$$

Then

$$\begin{aligned}
I = \int_0^{\pi} x\cos(x/2)\,dx &= \left[2x\sin(x/2)\right]_0^{\pi} - \int_0^{\pi} 2\sin(x/2)\,dx \\
&= \left[2x\sin(x/2) + 4\cos(x/2)\right]_0^{\pi} \\
&= (2\pi + 0) - (0 + 4) = \boxed{2\pi - 4}.
\end{aligned}$$

(d) Using the Washer Method, the volume is

$$V = \boxed{\int_0^\pi \pi\left(2^2 - (2 - x\cos(x/2))^2\right)\,dx}.$$



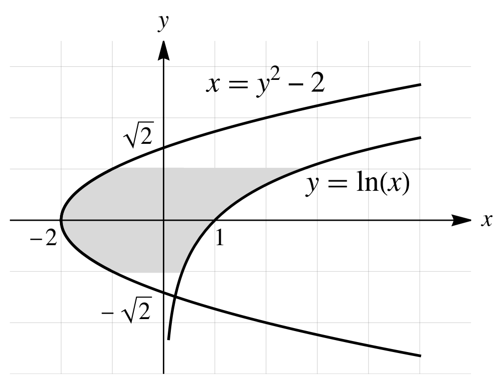
4. (15 pts) Consider the region $\mathcal{R}$ bounded by the curves $x = y^2 - 2$ and $y = \ln x$, between the lines $y = -1$ and $y = 1$.

   (a) Sketch and shade the region $\mathcal{R}$. Clearly label each function and all intercepts. (*Hint:* The functions intersect below the line $y = -1$.)

   (b) Evaluate an integral to find the area of region $\mathcal{R}$. (*Hint:* Integrate with respect to y.)

Solution:

(a)



(b) Note that $y = \ln x$ implies $x = e^y$, so the area is

$$\begin{aligned} A &= \int_{-1}^{1}\left(e^y - \left(y^2 - 2\right)\right)dy \\ &= \left[e^y - \frac{y^3}{3} + 2y\right]_{-1}^{1} \\ &= \left(e - \frac{1}{3} + 2\right) - \left(e^{-1} + \frac{1}{3} - 2\right) \\ &= \boxed{e - \frac{1}{e} + \frac{10}{3}}. \end{aligned}$$