1. (15 points) Consider $\mathbf{a} = 4\mathbf{i} - 5\mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - 7\mathbf{k}$, and $\mathbf{c} = \mathbf{j} + \mathbf{k}$.

   (a) For each of the following, either compute it, or, if it is not possible to compute, state "NOT POSSIBLE."

   i. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$

   ii. $(\mathbf{a} \cdot \mathbf{b}) \times \mathbf{c}$

   iii. $(\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c}$

   iv. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$

   (b) For which of the values that you found in (a) can you ascribe a *simple* geometric meaning in the context of the given vectors? What is this *simple* geometric meaning?

Solution:

(a) i. This is the scalar triple product, which is

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 4 & -5 & 3 \\ 2 & 0 & -7 \\ 0 & 1 & 1 \end{vmatrix} = 44.$$

ii. This is "NOT POSSIBLE" because the dot product produces a scalar, and we cannot take the cross product of a scalar and a vector.

iii. This is "NOT POSSIBLE" because the first dot product produces a scalar, and we cannot take the dot product of a scalar and a vector.

iv.

$$\begin{aligned} \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= \mathbf{a} \times \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -7 \\ 0 & 1 & 1 \end{vmatrix} \\ &= \langle 4, -5, 3 \rangle \times \langle 7, -2, 2 \rangle \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -5 & 3 \\ 7 & -2 & 2 \end{vmatrix} \\ &= \langle -4, 13, 27 \rangle. \end{aligned}$$

(b) The formula in (i) gives the volume of the parallelepiped generated by the three vectors, $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. (More generally, the absolute value of the scalar triple product is needed, but since the quantity is positive, we have the volume without the absolute value.)

2. (12 points) Consider the curves parameterized by $\mathbf{r_1}(t) = \langle 4t, 2 - t, t^2 + 1 \rangle$ and $\mathbf{r_2}(s) = \langle s^2, s^2 - 3, s \rangle$. Be sure to fully justify your reasoning for each of the following.

   (a) Do these two curves intersect? If so, at what point?

   (b) If these curves represent the locations of two particles at time, t (or s), do the two particles collide? If so, at what time and at what point? If not, explain why they do not collide.

Solution:

(a) We will consider $\mathbf{r_1}(t) = \mathbf{r_2}(s)$ and see if there are any solutions. This vector equation yields the system of equations

$$\begin{aligned} 4t &= s^2 \\ 2 - t &= s^2 - 3 \\ t^2 + 1 &= s. \end{aligned}$$

Since $s^2 = 4t$, the second equation becomes $2 - t = 4t - 3$. This linear equation has solution $t = 1$. From the first equation, this gives $s = \pm 2$. So, there are two possibilities. Using the third equation, we have equality for $t = 1$ and $s = 2$, but not $t = 1$ and $s = -2$. Then, we see that

$$\mathbf{r_1}(1) = \langle 4, 1, 2 \rangle = \mathbf{r_2}(2),$$

so the two curves intersect at the point $(4, 1, 2)$.

(b) The two particles do not collide. From (a), we see that the only point that both particles will pass through is $(4, 1, 2)$, but they do so at different times.

3. (23 points) Consider the point $P(2, 4, 2)$ and the line L_1 given by

$$x = 1 + t, \qquad y = 3 \qquad z = t.$$

(a) Determine the coordinates of the point, Q, on L_1 when $t = 0$.

(b) Find the vector projection of $\mathbf{u} = \langle 2, 0, 1 \rangle$ onto $\overline{PQ}$.

(c) Find the symmetric equations of the line, L_2, which passes through P and Q.

(d) Find the acute angle between lines L_1 and L_2.

Solution:

(a) $Q(1, 3, 0)$.

(b) We have $\overrightarrow{PQ} = \langle 1 - 2, 3 - 4, 0 - 2 \rangle = \langle -1, -1, -2 \rangle$. So, the vector projection is given by

$$\frac{\mathbf{u} \cdot \overrightarrow{PQ}}{||\overrightarrow{PQ}||^2} \overrightarrow{PQ} = -\frac{4}{6} \langle -1, -1, -2 \rangle = \langle \frac{2}{3}, \frac{2}{3}, \frac{4}{3} \rangle.$$

(c) A direction vector of L_2 is given by $\overrightarrow{PQ} = \langle 1 - 2, 3 - 4, 2 - 0 \rangle = \langle -1, -1, -2 \rangle$. If we use the point Q, then we have

$$x = 1 - t, \qquad y = 3 - t \qquad z = -2t.$$

Thus, symmetric equations for L_2 are

$$1 - x = 3 - y = \frac{z}{-2} \qquad \text{or} \qquad x - 1 = y - 3 = \frac{z}{2}.$$

(d) Direction vectors for these two lines are $\mathbf{a} = \langle 1, 0, 1 \rangle$ for L_1 and $\mathbf{b} = \langle -1, -1, -2 \rangle$ for L_2. The angle between these two vectors is

$$\theta = \arccos\left(\frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{a}||||\mathbf{b}||} \right) = \arccos\left(\frac{-3}{\sqrt{6}\sqrt{2}} \right) = \arccos\left(-\frac{\sqrt{3}}{2} \right) = \frac{5\pi}{6}.$$

So, the acute angle between these two lines is $\pi/6$ (or 30°).

4. (22 points) Consider the surface given by

$$x^2 + cy^2 + z^2 - 8z = 0$$

where c is a constant.

(a) i. For which value(s) of c will this surface be a sphere?
    ii. When this surface is a sphere, what is its center and radius?

(b) For which value(s) of c, if any, will this surface be:

(i) an ellipsoid? (ii) a cone? (iii) a hyperboloid of one sheet? (iv) a hyperboloid of two sheets?

(v) an elliptic paraboloid? (vi) a hyperbolic paraboloid? (vii) a cylinder?

(c) Suppose $c = 2$. Describe the intersection of this surface with the yz-plane with an equation on the yz-plane.

Solution:

First, we observe that it is useful to complete the square:

$$x^2 + cy^2 + (z-4)^2 = 4^2.$$

(a) This surface will be a sphere when $c = 1$. The center will be $(0, 0, 4)$ and the radius will be 4.

(b) i. $c > 0$.
    ii. None.
    iii. $c < 0$.
    iv. None.
    v. None.
    vi. None.
    vii. $c = 0$.

(c) The yz-plane is given by $x = 0$. So, plugging $x = 0$ and $c = 2$ into our equation, we have

$$2y^2 + (z-4)^2 = 4^2.$$

5. (28 points) A squirrel runs up a tree along a path with the squirrel's position given by $\mathbf{r}(t) = 4\sin t\mathbf{i} - 4\cos t\mathbf{j} + 3t\mathbf{k}$, where t is the number of seconds since the squirrel started running.

(a) Determine the rate of change of the direction (magnitude of the normal component of acceleration) of the squirrel for an arbitrary time t. .

(b) Determine the curvature of the squirrel's path for an arbitrary time t.

(c) How far has the squirrel traveled from $t = 0$ to $t = 2\pi$? (We want the length of the path traveled, not the squirrel's displacement.)

(d) Find the equation of the osculating plane (formed by $\mathbf{T}$ and $\mathbf{N}$) when $t = \pi/2$. Your final answer should be in the form $ax + by + cz + d = 0$.

Solution: It is useful for us to first note that we have the following:

$$\mathbf{r}'(t) = \langle 4\cos t, 4\sin t, 3\rangle$$
$$||\mathbf{r}'(t)|| = \sqrt{16\cos^2 t + 16\sin^2 t + 9} = 5$$

$$\begin{aligned}
\mathbf{r}''(t) &= \langle -4\sin t, -4\cos t, 0\rangle \\
\mathbf{T}(t) &= \frac{\mathbf{r}'(t)}{||\mathbf{r}'(t)||} = \left\langle \frac{4}{5}\cos t, \frac{4}{5}\sin t, \frac{3}{5}\right\rangle \\
\mathbf{T}'(t) &= \left\langle -\frac{4}{5}\sin t, \frac{4}{5}\cos t, 0\right\rangle \\
\mathbf{N}(t) &= \frac{\mathbf{T}'(t)}{||\mathbf{T}'(t)||} = \langle -\sin t, \cos t, 0\rangle.
\end{aligned}$$

(a) We see that

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4\cos t & 4\sin t & 3 \\ -4\sin t & 4\cos t & 0 \end{vmatrix} = \langle -12\cos t, -12\sin t, 16\rangle$$

. So,

$$a_{\mathbf{N}} = \frac{||\mathbf{r}'(t) \times \mathbf{r}''(t)||}{||\mathbf{r}'(t)||} = \frac{\sqrt{144\cos^2 t + 144\sin^2 t + 256}}{5} = \frac{\sqrt{400}}{5} = 4.$$

(b) We have $a_{\mathbf{N}} = \kappa v^2$. Since $a_{\mathbf{N}} = 4$ and $v = ||\mathbf{r}'(t)|| = 5$, then we have

$$\kappa = \frac{a_{\mathbf{N}}}{v^2} = \frac{4}{25}.$$

(c) We have

$$L = \int_0^{2\pi} ||\mathbf{r}'(t)||\, dt = \int_0^{2\pi} 5\, dt = 10\pi.$$

(d) We need a point and a normal vector of this plane. We note that the normal vector of the osculating plane is the binormal vector given by

$$\mathbf{B}\left(\frac{\pi}{2}\right) = \mathbf{T}\left(\frac{\pi}{2}\right) \times \mathbf{N}\left(\frac{\pi}{2}\right) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & \frac{4}{5} & \frac{3}{5} \\ -1 & 0 & 0 \end{vmatrix} = \left\langle 0, -\frac{3}{5}, \frac{4}{5}\right\rangle.$$

A point on this plane is given by $\mathbf{r}\left(\frac{\pi}{2}\right) = \left\langle 4, 0, \frac{3\pi}{2}\right\rangle$. So, the equation of the plane is given by

$$\begin{aligned}
\left\langle 0, -\frac{3}{5}, \frac{4}{5}\right\rangle \cdot \left\langle x-4, y, z-\frac{3\pi}{2}\right\rangle &= 0 \\
-\frac{3}{5}y + \frac{4}{5}\left(z - \frac{3\pi}{2}\right) &= 0 \\
3y - 4z + 6\pi &= 0 \text{ (or equivalent).}
\end{aligned}$$