

- This exam is worth 100 points and has 6 problems.
- Show all work and simplify your answers! Answers with no justification will receive no points unless otherwise noted.
- Begin each problem on a new page.
- **DO NOT LEAVE THE EXAM UNTIL YOU HAVE SATISFACTORILY SCANNED AND UPLOADED YOUR EXAM TO GRADESCOPE.**
- You are taking this exam in a proctored and honor code enforced environment. No calculators, cell phones, or other electronic devices or the internet are permitted during the exam. You are allowed one 8.5"× 11" crib sheet with writing on one side.

0. At the top of the first page that you will be scanning and uploading to Gradescope, write the following statement and sign your name to it: "I will abide by the CU Boulder Honor Code on this exam." **FAILURE TO INCLUDE THIS STATEMENT AND YOUR SIGNATURE MAY RESULT IN A PENALTY.**

1. [2360/021225 (10 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given. Please write your answers in a single column separate from any work you do to arrive at the answer.

- All solutions of the differential equation $\frac{dz}{dx} = -1 - x^4 - z^4$ approach $-\infty$ as $x \rightarrow \infty$ regardless of the initial condition.
- The equation $x' = (t - 3)(x + 2)^2(x - 1)$ has three equilibrium solutions.
- There exists a single real value of a that makes $a(\sin^2 t)y'' + (e^t - ay^{(4)})y' - y + a = 0$ a linear, homogeneous, separable differential equation.
- If L is a linear operator such that $L(\vec{u}) = 2t$ and $L(\vec{w}) = -t$, then $\vec{u} + 2\vec{w}$ is a solution to $L(\vec{y}) = 0$.
- $y^4 + 16y + x^4 - 8x^2 = 1$ is the implicit solution of $y' = \frac{4 + y^3}{4x - x^3}$ passing through $(2, 1)$.

2. [2360/021225 (19 pts)] A simple model to describe paying off the debt on a credit card is given by $A' = 0.2A - 600$, $A(0) = A_0$ where $A(t)$ (t in years) is the amount of the debt (dollars).

- (2 pts) What are the interest rate (%) and the yearly payment?
- (3 pts) Find the equilibrium solution. Interpret, by writing a sentence or two, the meaning of the equilibrium solution with regard to paying off the loan.
- (14 pts) If the initial amount of debt is \$2000, can the credit card debt be paid off in a finite amount of time? If so, find that time. If not, explain why not.

3. [2360/021225 (29 pts)] Consider the initial value problem $(t^2 + 4) \frac{dy}{dt} + 2ty = 4t$, $y(0) = 2$.

- (6 pts) Does Picard's Theorem guarantee that a unique solution to the IVP exists? Justify your answer.
- (8 pts) You are told to estimate $y(0.5)$ using 10 iterations of Euler's Method.
 - (2 pts) What is the step size, h ?
 - (3 pts) Find y_1 , the first output of Euler's method using the step size you found in part (i).
 - (3 pts) Will the estimate, y_1 , from part (ii) be the same if you use a different step size? Explain briefly.
- (15 pts) Use variation of parameters (Euler-Lagrange Two Step Method) to solve the IVP.

4. [2360/021225 (12 pts)] A 1000 kiloliter (kL) holding pond at a wastewater treatment plant is half full. The water in the pond initially contains 7 grams of dissolved heavy metal contaminants. Water having $(2 + \cos t)$ grams/kL of the heavy metals flows into the pond from a mining operation at 3 kL/day. In addition, a mountain stream containing 4 grams/kL of the heavy metals empties into the pond at 2 kL/day. The well-mixed contaminated water exits the holding pond at 5 kL/day.

- (9 pts) Write, but **DO NOT SOLVE**, an initial value problem that governs the amount, $c(t)$ (in grams), of the heavy metal contaminants in the pond at any time. Be sure to simplify your answer.
- (3 pts) Provide the largest interval over which the solution to the IVP is valid. Again, do not solve the IVP but do provide a brief reason for your answer.

5. [2360/021225 (15 pts)] The population of Wookiees, w (in thousands of animals), on the planet Kashyyyk is governed by the differential equation $dw/dt = (10 - w)w - 25$, where t is measured in decades. Chewbacca leaves the planet when $t = 1$ at which time there are 4000 Wookiees alive ($w = 4$). He returns several decades later to find out that the Wookiees went extinct while he was gone. Although brokenhearted, he wants to know the time, t_e , when the extinction occurred. Please find this time for him (and your grader).
6. [2360/021225 (15 pts)] No work needs to be shown and no partial credit will be given on this problem. The figure below shows the phase plane for the system

$$x' = x(4 - 2x - y)$$

$$y' = y(6 - 2x - 3y)$$

- (a) (2 pts) What do the solid lines represent?
- (b) (2 pts) What do the dashed lines represent?
- (c) (8 pts) Find all equilibrium solutions and determine their stability.
- (d) (3 pts) For each of the following initial conditions, find the limiting values (long term behavior) of x and y , that is, what are $\lim_{t \rightarrow \infty} x(t)$ and $\lim_{t \rightarrow \infty} y(t)$? Give your answers as an ordered pair (x, y) .

i. $(x(0), y(0)) = (\frac{1}{2}, 1)$

ii. $(x(0), y(0)) = (2, 0)$

iii. $(x(0), y(0)) = (3, 2)$

