1. (18 pts) Consider the function $f(x) = \dfrac{x+4}{\sqrt{x}}$ and answer the following:

   (a) What is the domain of $f(x)$? Give your answer in interval notation.

   (b) Does $f(x)$ have any horizontal asymptote(s)? To earn credit, use limit(s) to justify your answer. You may not use L'Hospital's rule or dominance of powers arguments.

   (c) Does $f(x)$ have any vertical asymptote(s)? To earn credit, use limit(s) to justify your answer. You may not use L'Hospital's rule or dominance of powers arguments.

Solution:

(a) In order to avoid taking the square root of a negative number, which would lead to a value in the denominator that is not a real number, we must have $x \geq 0$.

In order to avoid division by zero, we must have $x \neq 0$.

Therefore, the domain of $f(x)$ is the set of all values of x that satisfy both of the preceding requirements, which is $\boxed{(0, \infty)}$

(b)

$$\lim_{x\to\infty} \frac{x+4}{\sqrt{x}} = \lim_{x\to\infty} \left(\sqrt{x} + \frac{4}{\sqrt{x}}\right)$$

As $x \to \infty$, $\sqrt{x} \to \infty$ and $4/\sqrt{x} \to 0$. Therefore, the limit of their sum is infinity, so that

$$\lim_{x\to\infty} \left(\sqrt{x} + \frac{4}{\sqrt{x}}\right) = \boxed{\infty}$$

Note that $\lim\limits_{x\to-\infty} f(x)$ does not exist because the domain of f is $(0, \infty)$.

(c)

$$\lim_{x\to 0^+} \frac{x+4}{\sqrt{x}} \to \frac{4}{0^+} = \infty$$

Therefore, since at least one one-sided limit is infinite, $\boxed{f(x) \text{ has a vertical asymptote at } x = 0}$

Note that $\lim\limits_{x\to 0^-} f(x)$ does not exist because the domain of f is $(0, \infty)$.

2. (22 pts) Evaluate the following limits (be sure to show all justification. You may not use L'Hospital's rule or dominance of powers arguments.):

   (a) $\lim_{x\to\infty} \frac{\sqrt{5+x}-\sqrt{5}}{x}$

   (b) $\lim_{x\to 0} \frac{\sin(4x)\sin(7x)}{x^2}$

   (c) $\lim_{x\to 8} \frac{x-2x^{1/3}}{3x-12}$

Solution:

(a)

$$\begin{aligned}
\lim_{x\to\infty} \frac{\sqrt{5+x}-\sqrt{5}}{x} &= \lim_{x\to\infty} \frac{\sqrt{5+x}-\sqrt{5}}{x} \cdot \frac{\sqrt{5+x}+\sqrt{5}}{\sqrt{5+x}+\sqrt{5}} \\
&= \lim_{x\to\infty} \frac{(5+x)-5}{x(\sqrt{5+x}+\sqrt{5})} \\
&= \lim_{x\to\infty} \frac{x}{x(\sqrt{5+x}+\sqrt{5})} \\
&= \lim_{x\to\infty} \frac{1}{\sqrt{5+x}+\sqrt{5}}
\end{aligned}$$

The numerator is finite and as $x \to \infty$, the denominator approaches ∞. Therefore,

$$\lim_{x\to\infty} \frac{\sqrt{5+x}-\sqrt{5}}{x} = \boxed{0}$$

Alternative solution:

$$\begin{aligned}
\lim_{x\to\infty} \frac{\sqrt{5+x}-\sqrt{5}}{x} &= \lim_{x\to\infty} \frac{\sqrt{5+x}}{x} - \lim_{x\to\infty} \frac{\sqrt{5}}{x} \\
&= \lim_{x\to\infty} \frac{\sqrt{x\,(5/x+1)}}{x} - 0 \\
&= \lim_{x\to\infty} \frac{\sqrt{x}\,\sqrt{5/x+1}}{x} \\
&= \lim_{x\to\infty} \frac{\sqrt{5/x+1}}{\sqrt{x}}
\end{aligned}$$

As $x \to \infty$, the numerator approaches $\sqrt{0+1} = 1$ and the denominator approaches ∞. Therefore,

$$\lim_{x\to\infty} \frac{\sqrt{5+x}-\sqrt{5}}{x} = \boxed{0}$$

(b)

$$\begin{aligned}\lim_{x\to 0}\frac{\sin(4x)\sin(7x)}{x^2} &= \lim_{x\to 0}\frac{\sin(4x)}{x}\cdot\frac{\sin(7x)}{x}\\ &= \lim_{x\to 0}\frac{4\sin(4x)}{4x}\cdot\frac{7\sin(7x)}{7x}\\ &= 28\lim_{x\to 0}\frac{\sin(4x)}{4x}\cdot\lim_{x\to 0}\frac{\sin(7x)}{7x}\\ &= 28\cdot 1\cdot 1 = \boxed{28}\end{aligned}$$

(c) The function $\dfrac{x-2x^{1/3}}{3x-12}$ is the quotient of two continuous functions, and the denominator function does not equal zero at $x=8$. Therefore, $\dfrac{x-2x^{1/3}}{3x-12}$ is continuous at $x=8$, and it follows that the given limit can be evaluated using direct substitution.

$$\lim_{x\to 8}\frac{x-2x^{1/3}}{3x-12} = \frac{8-2\cdot 8^{1/3}}{3\cdot 8-12} = \frac{4}{12} = \boxed{\frac{1}{3}}$$

3. (28 points) The following problems are unrelated.

(a) A bird, perched on a sheer cliff, spots a beetle on the flat ground 100 feet away from the base of the cliff. The bird flies straight toward the beetle, snatches it up in its beak, and then runs 8 feet along the ground to join its flock. Suppose $30°$ is the angle between the bird's flight path and the ground. From the time the bird took flight, how far did the bird travel before joining its flock?

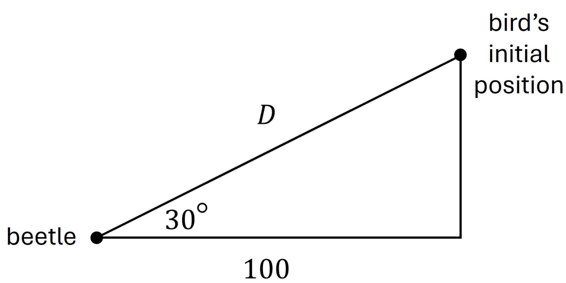
(b) Solve the equation $\cos(2t) = -\sin^2(t)$.

(c) Solve the inequality $3\cos(t) < \frac{3}{2}$ on the interval $[0, 2\pi)$.

(d) Sketch the graph of $g(x) = 2\sqrt{x+1}$. Be sure to label relevant intercept(s) on your graph.

Solution:

(a) Let D represent the straight-line distance between the bird's initial position and the beetle, as depicted below.



The preceding figure indicates that $\cos(30°) = \cos(\pi/6) = \dfrac{\text{adjacent}}{\text{hypotenuse}} = \dfrac{100}{D}$.

For the special angle $30° = \pi/6$ radians, the value of $\cos(30°) = \cos(\pi/6)$ is $\sqrt{3}/2$. Therefore,

$$\frac{\sqrt{3}}{2} = \frac{100}{D}$$

$$D = \frac{200}{\sqrt{3}}$$

Since D represents the distance the bird travels in the air, and it travels an additional 8 feet along the ground, the total distance traveled by the bird is $\boxed{200/\sqrt{3} + 8 \text{ feet}}$.

(b)

$$\begin{aligned}
\cos(2t) &= -\sin^2(t) \\
\cos^2(t) - \sin^2(t) &= -\sin^2(t) \qquad \text{(trig identity)} \\
\cos^2(t) &= 0 \\
\cos(t) &= 0 \\
t &= \boxed{\pi/2 + k\pi}\,, \text{ where } k \text{ is any integer}
\end{aligned}$$

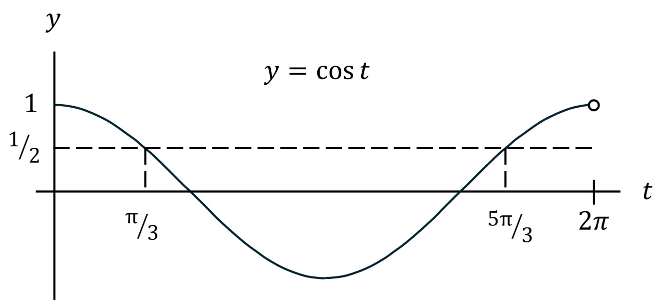
(c) Begin by solving the *equation* that corresponds to the given inequality.

$$3\cos t = \frac{3}{2}$$

$$\cos t = \frac{1}{2}$$

$$t = \pi/3, 5\pi/3 \quad \text{on the specified interval } [0, 2\pi)$$

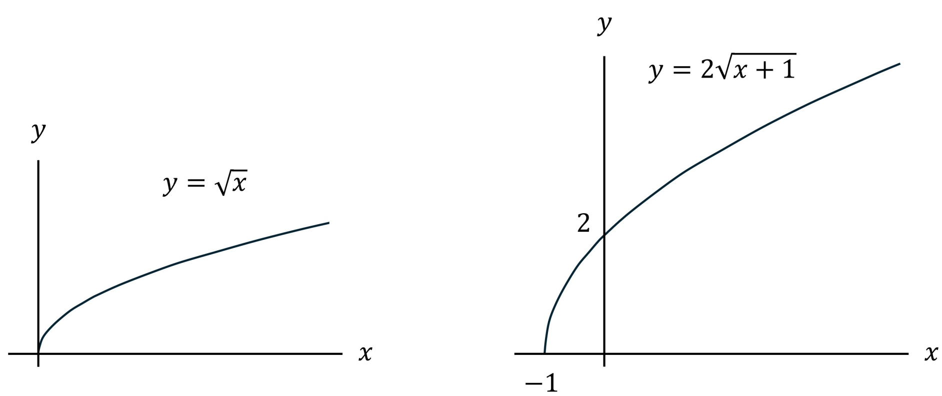
The graph of $y = \cos t$ on the interval $[0, 2\pi)$ is shown below, including the two values of t for which $\cos t = 1/2$.



The preceding graph indicates that the t interval on which $\cos t < 1/2$ is $\boxed{(\pi/3, 5\pi/3)}$

As an alternative method of solution, note that the values $t = \pi/3$ and $t = 5\pi/3$ serve as the boundary values of the subintervals $[0, \pi/3)$, $(\pi/3, 5\pi/3)$, and $(5\pi/3, 2\pi)$. The solution to the inequality could be obtained without drawing a graph by choosing a value of t from each of the subintervals, and determining whether or not that value of t satisfies the given inequality.

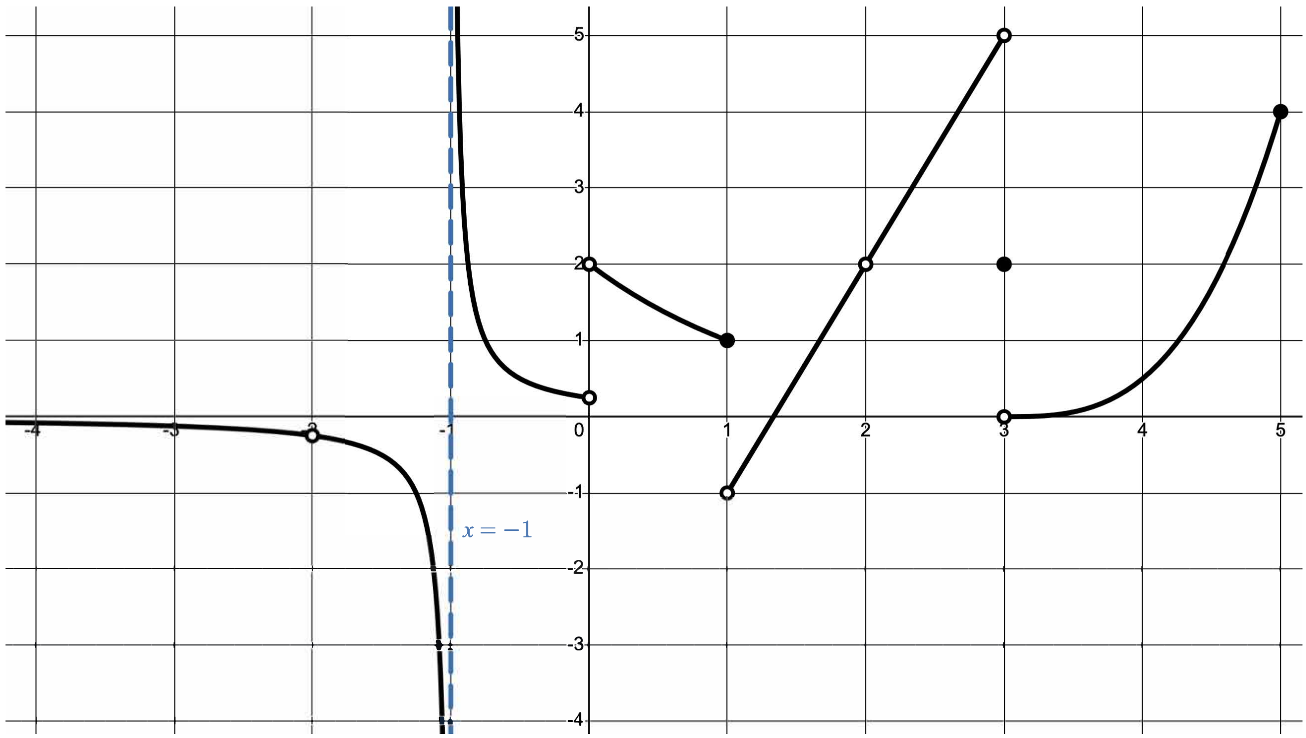
(d) The following figure depicts the base function curve $y = \sqrt{x}$ and the required transformed version of that curve. Replacing $\sqrt{x}$ with $\sqrt{x+1}$ serves to shift the base curve one unit to the left, and multiplying the result by two serves to stretch the resulting curve vertically by a factor of two. The x-intercept of $y = 2\sqrt{x+1}$ is the point $(-1, 0)$ and the y-intercept is the point $(0, 2)$, and those intercept values are included in the graph on the right, which is the final answer.



4. (18 points) Using the graph of $y = f(x)$ below, compute the following. If the limit does not exist, write DNE. Justification is not required for this problem.

(a) $\lim_{x \to -1^+} f(x)$ (b) $\lim_{x \to -1} f(x)$ (c) $\lim_{x \to 5^-} f(x)$ (d) $f(3)$

(e) $\lim_{x \to 2} f(x)$ (f) $\lim_{x \to -\infty} f(x)$ (g) $\lim_{x \to 1} |f(x)|$ (h) $\lim_{x \to 3^-} x f(x)$



Solution:

(a) $\lim_{x \to -1^+} f(x) = \boxed{\infty}$ because the function curve appears to increase without bound as it approaches the vertical dashed line at $x = -1$ from the right of that line.

(b) $\lim_{x \to -1} f(x)$ $\boxed{\text{does not exist}}$ because the function curve appears to increase without bound as it approaches the vertical dashed line at $x = -1$ from the right of that line, and it appears to decrease without bound as it approaches that line from the left.

(c) $\lim_{x \to 5^-} f(x) = \boxed{4}$ because as the value of x approaches 5 from the left, the function curve approaches a value of $y = 4$.

(d) The value of $f(3)$ is $\boxed{2}$ because the point $(3, 2)$ is included in the graph of $y = f(x)$. Note that the function value of 2 differs from the values of the left-hand and right-hand limits as x approaches 3.

(e) $\lim_{x \to 2} f(x) = \boxed{2}$ because the function curve approaches a value of $y = 2$ as x approaches a value of 2 from both the right and the left. Note that the function being undefined at $x = 2$ does not affect the limiting value.

(f) $\lim_{x\to-\infty} f(x) = \boxed{0}$ because the function curve appears to approach the line $y = 0$ (the x-axis) as the value of x increases without bound in the negative direction.

(g) $\lim_{x\to 1} |f(x)| = \boxed{1}$ because the function approaches a value of 1 as x approaches 1 from the left, and the function approaches a function value of -1 as x approaches 1 from the right. Therefore, since $|1| = |-1| = 1$, the absolute value of the function is approaching a value of 1 as x approaches 1 from both directions.

(h) $\lim_{x\to 3^-} xf(x) = \boxed{15}$ because

$$\begin{aligned}\lim_{x\to 3^-} xf(x) &= \lim_{x\to 3^-} x \cdot \lim_{x\to 3^-} f(x) \\ &= 3 \cdot \lim_{x\to 3^-} f(x)\end{aligned}$$

The graph indicates that the function approaches a value of 5 as x approaches 3 from the left. Therefore, the limiting value is $3 \cdot 5 = 15$.

5. (14 pts) The following problems are unrelated.

   (a) Is there a value of x such that $x^2 - \sqrt{x+1} = 4\sin(\pi x)$? Be sure to justify your answer and state any theorems you use.

   (b) What value of c makes the following function continuous? Be sure to justify your answer using the definition of continuity.

$$f(x) = \begin{cases} \frac{3}{4}x\,|x+1|, & \text{if } x < 1 \\ c, & \text{if } x = 1 \\ \frac{1}{4}\sqrt{x+3}+1, & \text{if } x > 1 \end{cases}$$

Solution:

(a) Let $g(x) = x^2 - \sqrt{x+1} - 4\sin(\pi x)$. The question in part (a) is equivalent to asking if there is a value of x such that $g(x) = 0$. The function $g(x)$ consists of a polynomial, a root function, and a sine function, all of which are continuous over their entire domains. Therefore, $g(x)$ is continuous on its entire domain since it is the sum of continuous functions. That being the case, use the **Intermediate Value Theorem**.

$$g(-1) = (-1)^2 - \sqrt{-1+1} - 4\sin(-\pi) = 1 - 0 - 0 = 1 > 0$$

$$g(0) = 0^2 - \sqrt{0+1} - 4\sin(0\cdot\pi) = 0 - 1 - 0 = -1 < 0$$

Therefore, since $g(x)$ is continuous on the interval $[-1, 0]$, $g(-1) > 0$, and $g(0) < 0$, the Intermediate Value Theorem indicates that there must be a value of $x = c$ on the interval $(-1, 0)$ such that $g(c) = 0$. This means that $c^2 - \sqrt{c+1} - 4\sin(\pi c) = 0$ and thus $x = c$ solves the equation $x^2 - \sqrt{x+1} = 4\sin(\pi x)$.

Note that pairs of values other than -1 and 0 could potentially be used in a similar manner. One such pair is 0 and 2.

(b) The function $\frac{3}{4}x\,|x+1|$ is continuous on the interval $(-\infty, 1)$ because polynomials and the absolute value function are continuous their domains. Likewise, the function $\frac{1}{4}\sqrt{x+3}+1$ is continuous on the interval $(1, \infty)$ because root functions are continuous their domains. So, $f(x)$ is continuous on $(-\infty, 1) \cup (1, \infty)$.

By definition, $f(x)$ is continuous at $x = 1$ if, and only if, the following holds:

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{3}{4}x\,|x+1| = \frac{3}{4}\cdot 1\cdot |1+1| = \frac{3}{2}$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \left(\frac{1}{4}\sqrt{x+3}+1\right) = \frac{1}{4}\sqrt{1+3}+1 = \frac{3}{2}$$

$$f(1) = c$$

Therefore, in order for $f(x)$ to be continuous at $x = 1$, and hence on $(-\infty, \infty)$, we must have $\boxed{c = 3/2}$