Write your name and section number below. This exam has 5 problems and is worth 100 points. On each problem, you must show all your work to receive credit on that problem. You are allowed to use one handwritten 3 x 5 inch notecard (front and back) on this exam. You are NOT allowed to use any other notes, books, calculators, or electronic devices.

After you finish the exam, go to the designated area of the room to scan and upload your exam to Gradescope. **Please be sure to match your work with the corresponding problem.** Do not leave the room until you verify that your exam has been correctly uploaded.

Name:

Instructor/Section (Dougherty-001, Mitchell-002, Becker-003):

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1. (31 points) For each of the following, **provide a short proof or justification**.

   (a) (8 points) If A and B are symmetric matrices of the same size, is AB symmetric?

   (b) (7 points) Let A be the matrix with permuted LU factorization given by the matrices

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \qquad U = \begin{pmatrix} 4 & -2 & 1 \\ 0 & -3 & 2 \\ 0 & 0 & 3 \end{pmatrix}$$

   What is the determinant of A?

   (c) (8 points) Recall that A and B are similar, written $A \sim B$, if there exists an invertible matrix S with $B = S^{-1}AS$. If $A \sim B$, is $det(A) = det(B)$?

   (d) (8 points) Define the trace of a square, $n \times n$ matrix A, to be $tr(A) = \sum_{i=1}^{n} a_{ii}$. For $n \times n$ matrices A and B, show that $tr(AB) = tr(BA)$.

Solution:

(a) No, it is not necessarily symmetric. Counterexample:

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$

(b) If $PA = LU$, then $det(PA) = det(P)det(A) = det(LU) = det(L)det(U)$. For the given matrices, $det(P) = -1$, $det(L) = 1$, and $det(U) = (4)(-3)(3) = -36$. Therefore, $det(A) = 36$.

(c) Yes, and to justify this, we can either do $det(B) = det(S^{-1}AS) = det(S^{-1})det(A)det(S) = 1/det(S)det(A)det(S) = det(A)$ using that $det(S^{-1}) = 1/det(S)$ (valid since S is invertible) and that $det(CD) = det(C)det(D)$.

Another way to see this is to note that $det(CD) = det(DC)$. Then, $det(B) = det(S^{-1}AS) = det(SS^{-1}A) = det(A)$ since $det(CD) = det(DC)$.

(d) We note that $(AB)_{ii} = \sum_{k=1}^{n} a_{ik}b_{ki}$ and $(BA)_{kk} = \sum_{i=1}^{n} b_{ki}a_{ik}$. Thus,

$$tr(AB) = \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}b_{ki} = \sum_{k=1}^{n}\sum_{i=1}^{n} b_{ki}a_{ik} = tr(BA)$$

2. (14 points)

Let A be the matrix with permuted LU factorization given by the matrices

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 4 & 2 & 4 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

and $\vec{b} = \begin{bmatrix} 0 \\ 6 \\ 5 \\ -2 \end{bmatrix}$

Find the solution $\vec{x}$ to $A\vec{x} = \vec{b}$. *To help check that your final answer is correct, you should have* $x_1 + x_2 + x_3 + x_4 = 1$.

Solution:

We solve $A\vec{x} = \vec{b}$ by first writing $PA\vec{x} = \tilde{\vec{b}}$ where $\tilde{\vec{b}} := P\vec{b} = \begin{bmatrix} 0 \\ -2 \\ 5 \\ 6 \end{bmatrix}$ (i.e., swapping 2nd and 4th rows, since that's the operation that P represents). Then we'll solve $L\underbrace{U\vec{x}}_{\vec{c}} = \tilde{\vec{b}}$ by first solving for $\vec{c}$ in $L\vec{c} = \tilde{\vec{b}}$ via forward substitution, and finally by solving for $\vec{x}$ in $U\vec{x} = \vec{c}$ via back substitution. Then forward substitution gives

$$\begin{aligned} c_1 &= \tilde{b}_1 & &\implies c_1 = 0 \\ c_2 &= \tilde{b}_2 - 2c_1 & &\implies c_2 = -2 \\ c_3 &= \tilde{b}_3 + 2c_2 & &\implies c_3 = 1 \\ c_4 &= \tilde{b}_4 - 2c_3 & &\implies c_4 = 4 \end{aligned}$$

so $\vec{c} = \begin{bmatrix} 0 \\ -2 \\ 1 \\ 4 \end{bmatrix}$. *Grading note: if you got* $\begin{bmatrix} 0 \\ 6 \\ 17 \\ -36 \end{bmatrix}$ *at this stage, you probably forgot to permute* $\vec{b}$.

Next we do back substitution on U and $\vec{c}$:

$$\begin{aligned} 2x_4 &= c_4 & &\implies x_4 = 3 \\ -x_3 &= c_3 & &\implies x_3 = -1 \\ 2x_2 &= c_2 & &\implies x_2 = -1 \\ 4x_1 &= c_1 - 2x_2 - 4x_3 - x_4 & &\implies x_1 = 1 \end{aligned}$$

and so the final answer is $\boxed{\vec{x} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 2 \end{bmatrix}.}$. *An answer of* $\begin{bmatrix} 20 \\ 3 \\ -17 \\ -18 \end{bmatrix}$ *is likely due to forgetting to permute* $\vec{b}$.

For reference, $LU = \begin{bmatrix} 4 & 2 & 4 & 1 \\ 8 & 6 & 8 & 2 \\ 0 & -4 & -1 & 0 \\ 0 & 0 & -2 & 2 \end{bmatrix}$ *and A is the same but with the 2nd and 4th rows swapped. Some students formed A and then solved for $\vec{x}$ via Gaussian Elimination, which is valid, but it misses the point of why we do LU factorizations in the first place, and it takes a lot more work.*

3. (15 points) Let $A = \begin{pmatrix} 1 & 0 & -2 \\ 3 & -1 & 0 \\ -2 & 1 & -3 \end{pmatrix}$. Find A^{-1}. Also **verify that you have indeed found** A^{-1}. *Hint: To help check your work, A^{-1} should only involve integers, no decimals/fractions.*

Solution:

We begin with the augmented matrix:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 3 & -1 & 0 & 0 & 1 & 0 \\ -2 & 1 & -3 & 0 & 0 & 1 \end{array}\right)$$

Multipy row 1 by -3 and add to row 2; multiply row 1 by 2 and add to row 3 to obtain:

$$\xrightarrow[R_3=R_3+2R_1]{R_2=R_2-3R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & -1 & 6 & -3 & 1 & 0 \\ 0 & 1 & -7 & 2 & 0 & 1 \end{array}\right)$$

Add row 2 to row 3:

$$\xrightarrow[R_3=R_3+R_2]{} \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & -1 & 6 & -3 & 1 & 0 \\ 0 & 0 & -1 & -1 & 1 & 1 \end{array}\right)$$

Multiply row 3 by 6 and add to row 2; multiply row 3 by -2 and add to row 1:

$$\xrightarrow[R_2=R_2+6R_3]{R_1=R_1-2R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -2 & -2 \\ 0 & -1 & 0 & -9 & 7 & 6 \\ 0 & 0 & -1 & -1 & 1 & 1 \end{array}\right)$$

For the last step, multiply row 2 and row 3 by -1:

$$\xrightarrow[R_3=-R_3]{R_2=-R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -2 & -2 \\ 0 & 1 & 0 & 9 & -7 & -6 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{array}\right)$$

To verify we have found A^{-1}, we multiply

$$AA^{-1} = \begin{bmatrix} 1 & 0 & -2 \\ 3 & -1 & 0 \\ -2 & 1 & -3 \end{bmatrix} \begin{bmatrix} 3 & -2 & -2 \\ 9 & -7 & -6 \\ 1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that you could also have calculated $A^{-1}A = I$. Either is fine.

4. (20 points) For which values of b and c does the system $x_1+x_2+bx_3 = 1$, $2x_1+3x_2-x_3 = -2$, and $3x_1 + 4x_2 + x_3 = c$ have (a) no solution? (b) exactly one solution? (c) infinitely many solutions? (Note: you do not have to find the solutions, just find the values of b and c and explain your work.)

Solution: We start with the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & b & 1 \\ 2 & 3 & -1 & -2 \\ 3 & 4 & 1 & c \end{array}\right]$$

Multiply row 1 by -2 and add to row 2; multiply row 1 by -3 and add to row 3:

$$\xrightarrow[R_3=R_3-3R_1]{R_2=R_2-2R_1} \left[\begin{array}{ccc|c} 1 & 1 & b & 1 \\ 0 & 1 & -1-2b & -4 \\ 0 & 1 & 1-3b & c-3 \end{array}\right]$$

Finally, multiply row 2 by -1 and add to row 3:

$$\xrightarrow[R_3=R_3-R_2]{} \left[\begin{array}{ccc|c} 1 & 1 & b & 1 \\ 0 & 1 & -1-2b & -4 \\ 0 & 0 & 2-b & c+1 \end{array}\right]$$

(a) The system is incompatible, and has no solution, if $b = 2$ and $c \neq -1$.

(b) The system has exactly one solution if $b \neq 2$ since then the original matrix has determinant equal to $2 - b$, which is not zero.

(c) The system has infinitely many solutions if $b = 2$ and $c = -1$ (since there is one free variable).

5. (20 points) Let $A = \begin{pmatrix} 2 & 1 & 4 & 11 \\ -2 & 0 & -2 & -8 \\ 4 & 3 & 10 & 25 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix}$. Find the general solution to $A\vec{x} = \vec{b}$.

Solution: We start with the augmented matrix:

$$\left(\begin{array}{cccc|c} 2 & 1 & 4 & 11 & 1 \\ -2 & 0 & -2 & -8 & -3 \\ 4 & 3 & 10 & 25 & 0 \end{array}\right)$$

Add row 1 to row 2 and multiply row 1 by -2 and add to row 3:

$$\xrightarrow[R_3=R_3-2R_1]{R_2=R_2+R_1} \left(\begin{array}{cccc|c} 2 & 1 & 4 & 11 & 1 \\ 0 & 1 & 2 & 3 & -2 \\ 0 & 1 & 2 & 3 & -2 \end{array}\right)$$

Multiply row 2 by -1 and add to row 3:

$$\xrightarrow[R_3=R_3-R_2]{} \left(\begin{array}{cccc|c} 2 & 1 & 4 & 11 & 1 \\ 0 & 1 & 2 & 3 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

We have two free variables, x_3 and x_4, and two basic variables, x_1 and x_2.

We set the free variables equal to real numbers s and t.

$$\begin{aligned} x_3 &= s \\ x_4 &= t \end{aligned}$$

Then row 2 gives us:

$$\begin{aligned} x_2 + 2x_3 + 3x_4 &= -2 \\ x_2 + 2s + 3t &= -2 \\ x_2 &= -2 - 2s - 3t \end{aligned}$$

Row 1 gives us:

$$\begin{aligned} 2x_1 + x_2 + 4x_3 + 11x_4 &= 1 \\ 2x_1 + (-2 - 2s - 3t) + 4s + 11t &= 1 \\ 2x_1 - 2 + 2s + 8t &= 1 \\ 2x_1 &= 3 - 2s - 8t \\ x_1 &= \frac{3}{2} - s - 4t \end{aligned}$$

So the solution is $\begin{pmatrix} \frac{3}{2} - s - 4t \\ -2 - 2s - 3t \\ s \\ t \end{pmatrix}$ which we can rewrite as

$$\begin{pmatrix} \frac{3}{2} \\ -2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ -2 \\ 1 \\ 0 \end{pmatrix} s + \begin{pmatrix} -4 \\ -3 \\ 0 \\ 1 \end{pmatrix} t$$